CHAPTER 9

INTRODUCTION TO CALCULUS

1. Two Problems with One Theme
2. Limits and Continuity
3. The Derivative
4. Applications of the Derivative
5. The Integral
6. Applications of the Integral
7. Introduction to Infinite Series
9.1 TWO PROBLEMS WITH ONE THEME

The Seventeenth Century is considered to be the dawn of modern mathematics. This remarkably prolific period culminated with the simultaneous and independent invention of calculus, as we know it today, by the English mathematician Sir Isaac Newton (1642 - 1727) and the German mathematician Gottfried Leibniz (1646 - 1716). Fifty years ago, the mathematics historian Howard Eves wrote, in his classic History of Mathematics book, that both Newton's and Leibniz's later years were marred by the conflict between the two of them over who should receive credit for the invention of calculus. Eves relates that although Newton made the discovery first, Leibniz was first to publish results. Eves goes on to say that, "If Leibniz was not as penetrating a mathematician as Newton, he was perhaps a broader one, and while inferior to his English rival as an analyst and mathematical physicist, he probably had a keener mathematical imagination and a superior instinct for mathematical form." It is certainly the case that Leibniz's notational scheme eventually won out and is the scheme we use today.

There are two main branches of calculus. **Differential Calculus** developed from a desire to find the slope of the tangent line to a curve at a point on the curve. **Integral Calculus** developed from a desire to find the area of a region bounded by curves. Closely related to this original function of integral calculus are the processes of finding volume and surface area of solids, length of a curve, force, mass, centroids and work. The operations of differential and integral calculus are inverse operations and in fact indefinite integration is often called anti-differentiation.

THE AREA PROBLEM

As you have seen in the geometry chapters earlier in this book, the Greeks knew how to find areas of polygons. As you saw in the **ESTIMATING II INVESTIGATION**, Archimedes used polygons inscribed in and circumscribed about a circle to approximate \( \pi \). His work involved first recognizing that for any circle, the ratio of the circumference to the diameter appeared to be a constant value slightly larger than 3. Then, by calculating the perimeter of both the inscribed and circumscribed polygons and dividing by the diameter of the circle, he was able to find upper and lower bounds for the ratio of the circumference to the diameter of the circle. Today we call this ratio \( \pi \).

The Greeks also realized that areas of regions with curved boundaries, such as a circle, could be approximated by inscribing in and/or circumscribing about the region various polygons such as triangles and rectangles.

Consider the region bounded by the graph of \( y = x^2 \), the x-axis, and the line \( x = 1 \) shown on the left below. Clearly the area of the region is less than 1. Closer examination also reveals that the area must be less than \( \frac{1}{2} \), because it must be less than the triangle with vertices (0, 0), (1, 0), and (1, 1).
Now consider approximating the area by four equal width rectangles circumscribing the region as shown on the right above. The heights of the rectangles are \( \frac{1}{16}, \frac{4}{16}, \frac{9}{16}, \text{ and } \frac{16}{16}, \) respectively. Thus the area of the region is less than \( \frac{1}{4} \left( \frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16} \right) = \frac{15}{32}. \)

Next suppose we double the number of circumscribed rectangles producing the picture below. Calculating the sum of the areas of these 8 circumscribed rectangles produces \( \frac{1}{8} \left( \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} + \frac{25}{64} + \frac{35}{64} + \frac{49}{64} + \frac{64}{64} \right) = \frac{51}{128}. \) Thus the area of the region is less than \( \frac{51}{128}. \)

It should be clear that by increasing the number of rectangles, we obtain a better approximation to the true area. If we call \( n \) the number of rectangles, then as \( n \) approaches \( \infty, \) which we write as \( n \to \infty, \) the approximation for the area approaches the true area. Consider the picture below.
Calculating the sum of the areas of these $n$ circumscribed rectangles produces

$$
\frac{1}{n} \left[ \left( \frac{1}{n} \right)^2 + \left( \frac{2}{n} \right)^2 + \left( \frac{3}{n} \right)^2 + \ldots + \left( \frac{(n-1)}{n} \right)^2 + \left( \frac{n}{n} \right)^2 \right] = \frac{1}{n} \left[ \frac{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2}{n^2} \right]
$$

Now we want to evaluate this expression for the sum of the areas of our $n$ rectangles as $n \to \infty$. Some notation and formulas which will facilitate our work in finding area will be developed at this time.

Before we return to the problem, some mention of notation is needed. Sums are often denoted more compactly using what is called \textbf{sigma notation}.

$$
\sum_{i=1}^{4} 3i = 3(1) + 3(2) + 3(3) + 3(4) = 30.
$$

Observe that $i$ takes on the integer values from 1 to 4. You substitute in 1, 2, 3, and 4 for $i$ in the $3i$ expression and then add up the results. In general, the algebraic properties apply to summation, as you would expect. Some examples are shown below.

$$
\sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1 \ n \text{ times} = n
$$

$$
\sum_{i=1}^{n} ki = k \sum_{i=1}^{n} i = k \left( 1 + 2 + 3 + \ldots + n \right), \ k \text{ a constant}
$$

$$
\sum_{i=1}^{n} (2i - 3)^2 = \sum_{i=1}^{n} (4i^2 - 12i + 9) = 4 \sum_{i=1}^{n} i^2 - 12 \sum_{i=1}^{n} i + \sum_{i=1}^{n} 9 = 4 \frac{n(n+1)(2n+1)}{6} - 12 \frac{n(n+1)}{2} + 9n
$$
Now to develop some summation formulas, first observe that \( n^2 - (n - 1)^2 = 2n - 1 \).
Since this equation is valid for any value of \( n \), we have:
\[
\begin{align*}
n^2 - (n - 1)^2 &= 2n - 1 \\
(n - 1)^2 - (n - 2)^2 &= 2(n - 1) - 1 \\
(n - 2)^2 - (n - 3)^2 &= 2(n - 2) - 1 \\
& \quad \vdots \\
2^2 - 1^2 &= 2(2) - 1 \\
1^2 - 0^2 &= 2(1) - 1
\end{align*}
\]
Summing both sides produces \( n^2 = \sum_{i=1}^{n} i \)  
Hence \( \sum_{i=1}^{n} i = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} \)

Next observe that \( n^3 - (n - 1)^3 = 3n^2 - 3n + 1 \) and as before,
\[
\begin{align*}
(n - 1)^3 - (n - 2)^3 &= 3(n - 1)^2 - 3(n - 1) + 1 \\
(n - 2)^3 - (n - 3)^3 &= 3(n - 2)^2 - 3(n - 2) + 1 \\
2^3 - 1^3 &= 3(2)^2 - 3(2) + 1 \\
1^3 - 0^3 &= 3(1)^2 - 3(1) + 1
\end{align*}
\]
Summing both sides produces \( n^3 = \sum_{i=1}^{n} i^2 - 3 \sum_{i=1}^{n} i + n \)
Substituting \( \frac{n(n+1)}{2} \) for \( \sum_{i=1}^{n} i \) and solving for \( \sum_{i=1}^{n} i^2 \) produces \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

Exercise: Show that \( \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \)

Other summation formulas can be similarly developed but these three will be sufficient for our work here. Now returning to our problem of finding the area under the graph of \( y = x^2 \) from 0 to 1, we need to find the limit of our rectangle areas formula as \( n \to \infty \). We write this as \( \lim_{n \to \infty} \frac{1}{n^3} \left[ 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \right] = \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \).
Using our formula we just developed we have \( A_0^1 = \lim_{n \to \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \).
Observe that for this rational function in \( n \), the numerator and denominator are the same degree (3). Thus, as we saw in section 7.2, the limit is \( \frac{2}{6} = \frac{1}{3} \) and we have \( A_0^1 = \frac{1}{3} \).
Now suppose we wanted to find the area under the graph of $y = x^2$ from 0 to some number $a$. Our work would look the same as before.

$$A_a^n = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{a}{n} \right)^2 = \lim_{n \to \infty} \frac{a^3}{n} \sum_{i=1}^{n} i^2 = \lim_{n \to \infty} \frac{a^3}{n} \frac{n(n+1)(2n+1)}{6} = \frac{a^3}{3}$$

**Note 1:** Observe that the area is a function of $a$. We could just as well express the area as a function of $x$ and say that the area from 0 to $x$ under $y = x^2$ equals $\frac{x^3}{3}$. From your previous work with calculus, you can see that the area from 0 to $x$ under the graph of the function is the anti derivative or indefinite integral of the function. This statement is true for any continuous function.

**Note 2:** You should also realize that if we wanted the area from 1 to 4, we could find $A_0^4 - A_0^1 = \frac{4^3}{3} - \frac{1^3}{3} = 21$. This foreshadows the process of definite integration.

**Note 3:** The preceding development, which foreshadowed integral calculus, originated over two millennia before differential calculus (which occurred in the Seventeenth Century). Today we teach differential calculus first.

You will have the opportunity to do more of this area work in the problem set for this section, but now we will consider the other fundamental calculus process, that of finding the slope of a curve at a point on the curve.

**THE TANGENT PROBLEM**

Consider the graph of $y = f(x) = x^2$ shown again below.
Chapter 9: Introduction to Calculus

The point $P(x, f(x))$ and a nearby point $Q(x + \Delta x, f(x + \Delta x))$ are shown ($\Delta x$ is called the change in $x$ and represents a small positive number). As you saw in geometry, the line through these two points is called a secant line. If we let $\Delta x \to 0$, then $Q \to P$ and the secant line becomes a tangent line. Symbolically, in terms of slope $m$, we write this as:

$$m_{\text{lim}} = \lim_{\Delta x \to 0} m_{\text{sec}} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If we want the slope at $x = 2$ then

$$m_{x=2} = \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(2 + \Delta x)^2 - (2)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{4 + 4\Delta x + \Delta x^2 - 4}{\Delta x} = \lim_{\Delta x \to 0} \frac{4\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} (4 + \Delta x) = 4$$

Now suppose we want the slope at $x = a$

$$m_{x=a} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(a + \Delta x)^2 - (a)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^2 + 2a\Delta x + \Delta x^2 - a^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2a\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} (2a + \Delta x) = 2a$$

Similarly to what we did in the area problem, the slope of the tangent at any point $(x, f(x))$ on the curve would be given by $2x$. This "formula" for the slope is derived from the original equation and hence is called a derivative. In general, $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is called the definition of the derivative, which is sometimes symbolized by $D_x f(x)$.

**Problem Set 9.1**

1. Write a report detailing the development of Integral Calculus.

2. Research Newton's and Leibniz's derivative notations and write a report detailing the development of Differential Calculus.

3. Use the techniques described in this section to find the area under $y = 2x + 1$ between $x = 0$ and $x = 4$. This method is, of course, more difficult than what you would normally do since you have a simple polygon. Compare your result with what you know the area to be.
9.2 LIMITS AND CONTINUITY

You have seen that the two primary problems of calculus both involved finding limits. It is now time to study the process of finding limits and to understand when limits do or do not exist. First we will examine limits of the type seen in the tangent problem. The definition of a limit for a function defined on intervals to the left and right of some number \( a \), is shown below.

\[
\lim_{{x \to a}} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon
\]

This highly symbolic statement is best explained with both a picture and some words.

To say that as \( x \to a \), \( f(x) \to L \) means that for any \( \varepsilon > 0 \), there exists a corresponding \( \delta > 0 \) such that when \( x \) is within \( \delta \) of \( a \), \( f(x) \) is within \( \varepsilon \) of \( L \). In other words, if we want to get within \( \varepsilon \) of \( L \), we can do so as long as we choose our \( x \)-values carefully, being sure to go no farther than \( \delta \) away from \( a \). Showing the rule of correspondence between \( \delta \) and \( \varepsilon \) constitutes proving that a limit exists and equals \( L \).

**Example 1:** To prove that \( \lim_{{x \to 2}} (3x - 2) = 4 \), we must show

\[
\forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - 2| < \delta \Rightarrow |3x - 2 - 4| < \varepsilon
\]

Observe that the right side of the implication can be rewritten as

\[
|3x - 2 - 4| < \varepsilon \iff |3x - 2| < \varepsilon
\]

Since the two intervals have the same center, to make the left interval imply the right interval, all we need do is make the radius of the left interval no more than the radius of the right interval. So \( \delta \leq \varepsilon/3 \) shows a rule of correspondence of \( \delta \) to any choice of \( \varepsilon \), and thus the limit value is correct.

**Example 2:** To prove that \( \lim_{{x \to 2}} (3x^2 - 2x) = 8 \), we must show

\[
\forall \varepsilon > 0, \exists \delta > 0 \ni 0 < |x - 2| < \delta \Rightarrow |3x^2 - 2x - 8| < \varepsilon
\]

Proceeding as in Example 1, \( |3x^2 - 2x - 8| < \varepsilon \iff |(3x^2 - 2x) - 8| < \varepsilon \)

\[
\iff |3x^2 - 2x| < \varepsilon \iff |3x^2 - 2x - 8| < \varepsilon
\]

and thus we might wish to choose \( \delta \leq \frac{\varepsilon}{|3x + 1|} \). This, however, is not a satisfactory correspondence, because \( \delta \) depends on both \( \varepsilon \) and \( x \). We can resolve this problem by realizing that whenever we find a \( \delta \) that works, any smaller \( \delta \) will also work. This means that we can, without loss of generality, place a maximum value on \( \delta \). Now, if in the problem at hand, we say \( \delta \leq 1 \), then \( |x - 2| < \delta \)
\[ |x - 2| < 1 \Rightarrow 1 < x < 3. \] The smallest value of \( \frac{\varepsilon}{|3x + 1|} \) on \([1,3]\) is \( \frac{\varepsilon}{10} \).

Thus we can express \( \delta \leq \min \left\{ 1, \frac{\varepsilon}{10} \right\} \) and our limit exists and is equal to 8.

Now we will examine the type of limits seen in the area problem.

\[
\lim_{x \to \infty} f(x) = L \iff \forall \varepsilon > 0, \exists n > 0, n \in \mathbb{N} \ni x > n \Rightarrow |f(x) - L| < \varepsilon
\]

To say that as \( x \to \infty \), \( f(x) \to L \) means that to any measure, \( \varepsilon > 0 \), there corresponds some positive integer \( n \) such that when \( x > n \), \( f(x) \) is within \( \varepsilon \) of \( L \).

**Example:** To prove that \( \lim_{x \to \infty} \frac{2x + 3}{3x - 4} = \frac{2}{3} \) we must find \( n \) so that

\[
x > N \Rightarrow \left| \frac{2x + 3}{3x - 4} - \frac{2}{3} \right| < \varepsilon
\]

\[
\left| \frac{2x + 3}{3x - 4} - \frac{2}{3} \right| < \varepsilon \iff \left| \frac{6x + 9 - 6x + 8}{3(3x - 4)} \right| < \varepsilon \iff \left| \frac{17}{3(3x - 4)} \right| < \varepsilon
\]

\[
\left| \frac{3(3x - 4)}{17} \right| > \frac{1}{\varepsilon} \iff |3x - 4| > \frac{17}{3\varepsilon} \iff 3x - 4 > \frac{17}{3\varepsilon} \quad (\text{Since } x \to \infty)
\]

\[
3x > \frac{17}{3\varepsilon} + 4 = \frac{17 + 12\varepsilon}{3\varepsilon} \iff x > \frac{17 + 12\varepsilon}{9\varepsilon}
\]

Now since for any \( \varepsilon \) value, there would be an integer, \( n > \frac{17 + 12\varepsilon}{9\varepsilon} \), for any \( \varepsilon \) there corresponds an integer \( n \) and our limit is correct.

As you might guess, there are rules to apply when working with limit problems. Happily, the rules are just what you would do if you had no rules. If \( A \) and \( B \) are functions, then

\[
\text{Lim } (A + B) = \text{Lim } A + \text{Lim } B
\]
\[
\text{Lim } (A - B) = \text{Lim } A - \text{Lim } B
\]
\[
\text{Lim } (A \cdot B) = \text{Lim } A \cdot \text{Lim } B
\]
\[
\text{Lim } (A/B) = \text{Lim } A / \text{Lim } B
\]
\[
\text{Lim } kA = k \text{ Lim } A
\]

In words, the limit of a sum, difference, product or quotient of functions equals the sums, differences, products or quotients of the limits of the functions. Also, the limit of a constant times a function equals the constant times the limit of the function. As you have seen, finding limits is a tool of both differential and integral calculus. There is much
more that we could do with limits, but this is enough for now. You will have the opportunity to do more work with limits in the problem set for this section.

CONTINUITY

An important concept related to limits is that of continuity. Basically, a function is continuous on an interval when it can be drawn in the interval without lifting your pencil from the paper. A formal definition for continuity at a point is shown below.

A function \( f(x) \) is said to be continuous at \( x = c \) if

\[
(1) \ f(c) \text{ exists} \quad (2) \ \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad (3) \ \lim_{x \to c} f(x) = f(c).
\]

To illustrate this definition of continuity, consider the three examples below.

\textbf{Example 1:} \( f(x) = \frac{1}{x-2} \) is discontinuous at \( x = 2 \) because \( f(2) \text{ DNE (does not exist)} \).

\textbf{Example 2:} \( f(x) = \begin{cases} \frac{1}{x-2}, & x \neq 2 \\ \frac{1}{3}, & x = 2 \end{cases} \) is discontinuous at \( x = 2 \) because \( \lim_{x \to 2} f(x) \text{ DNE} \).

\textbf{Example 3:} \( f(x) = \begin{cases} \frac{x^2 - 4}{x-2}, & x \neq 2 \\ \frac{1}{3}, & x = 2 \end{cases} \) is discontinuous at \( x = 2 \) because \( \lim_{x \to 2} f(x) \neq f(2) \).

The discontinuity exhibited by a function such as \( f(x) = \frac{x^2 - 4}{x - 2} \) is referred to as a point or removable discontinuity. Observe that this function can be made continuous for all \( x \) by defining \( f(2) = 4 \) and thus "filling in the hole" in the graph. Functions can also have an infinite discontinuity by going to \( \infty \); or a jump discontinuity by "jumping" from one part of the graph to another. As examples, \( f(x) = \frac{1}{x} \) exhibits an infinite discontinuity at \( x = 0 \), and \( f(x) = [[x]] \) (the greatest integer function) exhibits jump discontinuity at each integer value of \( x \). Functions also can be what is called continuous from the right if \( \lim_{x \to a^+} f(x) = f(a) \) or from the left if \( \lim_{x \to a^-} f(x) = f(a) \). As examples, \( f(x) = \sqrt{x} \) is continuous from the right but not from the left at \( x = 0 \), and \( f(x) = \sqrt{-x} \) is continuous from the left but not from the right at \( x = 0 \).

Now, recall that the notion of a derivative was introduced in the last section. Derivatives will be examined in detail in the next section. At this time, however, it is appropriate to mention an important relationship that exists between derivatives and continuity. You saw that a derivative \( D_x f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \). This formula says that you compute the slope of a secant line through two points on a curve and then take the limit as one point approaches the other point. By labeling the points (P and Q) in a different manner, as \((x, f(x))\) and \((c, f(c))\), then \( D_x f(x) \) at \((c, f(c)) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \). As \( x \to c \), \( P \to Q \) and we have the slope of the tangent line at \((c, f(c))\). This concept is pictured below.
Now if the limit exists, then, since the denominator → 0, the numerator must also → 0. Thus \( \lim_{x \to c} [f(x) - f(c)] = 0 \Rightarrow \lim_{x \to c} f(x) = \lim_{x \to c} f(c) = f(c) \). Hence f(x) is continuous at c.

So we see that \textbf{Differentiability at a Point } \Rightarrow \textbf{Continuity at the Point}

\textbf{Note:} The converse of the above statement is not true. For example the absolute value function is continuous at 0, but does not have a derivative there.

We conclude this section with some theorems related to continuity that are fairly obvious, but difficult to formally prove.

1. The sum, difference, product and quotient (except where the denominator = 0) of continuous functions are continuous.
2. A function continuous on \([a, b]\) has a minimum and a maximum value in the interval.
3. If \( f(x) \) is continuous on \([a, b]\) and \( n \) is any value between \( f(a) \) and \( f(b) \), then there exists at least one value \( c \) between \( a \) and \( b \), such that \( f(c) = n \). [The function takes on all values in the interval \([f(a), f(b)]\).

\textbf{Note:} This statement is often called the \textit{Intermediate Value Theorem - IVT}.]

You will see the concept of continuity applied many times in subsequent sections. You will also see a few problems dealing with continuity in the problem set for this section.

\textbf{Problem Set 9.2}

1. Evaluate the following limits or explain why they don't exist.

(A) \( \lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8} \)  (B) \( \lim_{x \to 1} \frac{1 - 4x}{1 - x} \)  (C) \( \lim_{x \to \infty} \frac{2x^3 - 3x + 4}{5 - 6x^3} \)  (D) \( \lim_{x \to \infty} \left( \sqrt{x^2 + x} - \sqrt{x^2 + 7} \right) \)

(E) \( \lim_{x \to -\infty} \left( \sqrt{x^2 + x} - \sqrt{x^2 + 7} \right) \)  (F) \( \lim_{x \to 3^+} \frac{[x]^7 - 9}{x^2 - 9} \)  (G) \( \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} \)

2. For \( x \neq 5 \), \( f(x) = \frac{x^2 - 3x - 10}{x - 5} \). What value should be assigned to \( f(5) \) to make \( f(x) \) continuous on \((-\infty, \infty)\)?

3. If \( f(x) \) is continuous on \((-\infty, \infty)\) and \( f(a) > 0 \) and \( f(b) < 0 \), what do we know must occur somewhere between \( a \) and \( b \)?
9.3 THE DERIVATIVE

Even though the creation of integral calculus preceded that of differential calculus by over two thousand years, it has become standard to study differential calculus first. The tangent problem mentioned in section 9.1 gave rise to differential calculus.

As you saw in section 9.1, the formula for the slope of the tangent line to a curve at a point $x$ on the curve is calculated by $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$. As mentioned previously, this formula is called the derivative of $f(x)$ because it is derived from the original equation. A derivative of a function $f(x)$ is usually denoted by $f'(x)$ or $D_x f(x)$. If $y = f(x)$, then other symbols for the derivative are $D_y y'$, or $\frac{dy}{dx}$. This last symbolic form, proposed and used by Leibniz, is motivated by the definition of a derivative. Observe that the numerator in the limit expression can be considered to be the change in $y$. Thus $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The limit process to calculate a derivative can be tedious. Consequently, rules for differentiation have been developed to facilitate finding derivatives of various functions. These rules are developed by applying the limit process for general functions. Frequently, in this work an $h$ is used in place of the $\Delta x$.

THE POWER RULE

Consider $y = f(x) = x^n$ where $n$ is a whole number.

$$y' = f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

which by the Binomial Theorem

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1}$$

$$= nx^{n-1}$$

This result is called the **Power Rule**.

**Exercise:** Polynomials are collections of terms of the form $kx^n$ where $n$ is a whole number and $K$ is some constant. Use the limit definition of a derivative to convince yourself that derivative of $4x^2 + 8x - 12$ equals four times the derivative of $x^2 + 2x - 3$ and that the derivative of $2x^2 + 3x - 8$ equals the derivative of $x^2 + 2x - 3$ plus the derivative of $x^2 + x - 5$.

Then show in general that $D_x kf(x) = kD_x f(x)$, and that $D_x [f(x) + g(x)] = D_x f(x) + D_x g(x)$.

Consequently you are now able to differentiate all polynomial functions.

THE PRODUCT RULE
In the exercise above you saw that the derivative of a sum equals the sum of the derivatives. Previously you saw that the limit of a product equals the product of the limits. Perhaps this prompts the question in your mind, "Is the derivative of a product equal to the product of the derivatives?". Consider \( f(x) = 2x + 1 \) and \( g(x) = 3x - 2 \). Since \( f'(x) = 2 \) and \( g'(x) = 3 \), \( f'(x)g'(x) = 6 \). But \( f(x)g(x) = (2x + 1)(3x - 2) = 6x^2 - x - 2 \) and thus \( D_x[f(x)g(x)] = 12x - 1 \). Consequently \( D_x[f(x)g(x)] \neq f'(x)g'(x) \), and we must go back to the limit definition to develop the correct rule.

\[
\frac{d}{dx} [f(x)g(x)] = \lim_{h \to 0} \frac{(x + h)g(x + h) - f(x)g(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x + h) + f(x)g(x + h) - f(x)g(x)}{h}
\]
\[
= \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h}g(x + h) + \frac{g(x + h) - g(x)}{h}f(x) \right]
\]
\[
= f'(x)g(x) + g'(x)f(x)
\]

This result is called the **Product Rule**.

**THE QUOTIENT RULE**

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}
\]

**Exercise:** Develop the **Quotient Rule**

**Note:** Having the Quotient Rule allows us to now differentiate all rational functions.

**Exercise:** Show that the Power Rule holds for negative powers by writing \( y = x^{-n} \) as \( y = \frac{1}{x^n} \) and using the Quotient Rule just developed.

**THE EXTENDED POWER RULE**
Now in the Product Rule, what if \( f(x) \) and \( g(x) \) are the same function?

Then
\[
\frac{d}{dx} f^2(x) = \frac{d}{dx} [f(x)f(x)] = f'(x)f(x) + f'(x)f(x) = 2f(x)f'(x)
\]

\[
\frac{d}{dx} f^3(x) = \frac{d}{dx} [f^2(x)f(x)] = [2f(x)f'(x)f(x) + f'(x)f^2(x)]
\]

\[
= 3 f^2(x)f'(x)
\]

Observing and continuing this pattern produces
\[
\frac{d}{dx} f^n(x) = n f^{n-1}(x)f'(x) \quad n \in \mathbb{I}
\]

which is called the **Extended Power Rule** (EPR).

**THE CHAIN RULE**

The EPR foreshadows a very important derivation rule called the **Chain Rule**. Recall previously studying the composition of functions. We are now ready to differentiate composite functions of the form \( y = f[g(x)] \).

Let \( f(x) = 2x - 1 \) and \( g(x) = 3x + 2 \)
Then \( f[g(x)] = 2(3x + 2) - 1 = 6x + 3 \)

Observe that \( \frac{d}{dx} f(x) = 2, \frac{d}{dx} g(x) = 3 \) and \( \frac{d}{dx} f[g(x)] = 6 \)

Now let \( f(x) = 3x^2 - 1 \) and \( g(x) = 2x^3 + 3 \)
Then \( f[g(x)] = 3(2x^3 + 3)^2 - 1 = 3(4x^6 + 12x^3 + 9) - 1 = 12x^6 + 36x^3 + 26 \)

Observe that \( \frac{d}{dx} f(x) = 6x, \frac{d}{dx} g(x) = 6x^2 \) and \( \frac{d}{dx} f[g(x)] = 72x^5 + 108x^2 \).

With considerable inspired algebra, you might determine that the rule which produces the correct result for both of these problems is:

\[
\frac{d}{dx} f[g(x)] = f'[g(x)]g'(x)
\]

For example in the second example \( f'(x) = 6x \) and \( g(x) = 2x^3 + 3 \Rightarrow f'[g(x)] = 6(2x^3 + 3) \)
and \( g'(x) = 6x^2 \). Thus \( f'[g(x)] \cdot g'(x) = 6(2x^3 + 3)(6x^2) = 72x^5 + 108 x^2 \)

Now how should we remember what to do? Suppose \( y = f(u) \) and \( u = g(x) \). Then \( y = f(u) = f[g(x)] \). Now using Leibnitz's notation for derivatives, \( \frac{dy}{du} = f'(u), \frac{du}{dx} = g'(x) \)
and \( \frac{dy}{dx} = f'[g(x)]g'(x) = f'(u)g'(x) = \frac{dy}{du} \frac{du}{dx} \).

The derivatives "hook" together like links of a chain (which suggest where the name came from). A formal proof of the chain rule can be found in any standard calculus book.

**Note:** The extended Power Rule seen earlier is a special case of the Chain Rule.

\[
\frac{d}{dx} f^n(x) = n f^{n-1}(x) f'(x).
\]

The Chain Rule is a powerful derivative rule which allows you to easily differentiate complicated nested functions. For example consider differentiating \( f[g[h(x)]] \). You "cook" this problem like a conventional oven from outside in.
The derivative is $f'[g(h(x))] \cdot g'[h(x)] \cdot h'(x)$.

**IMPLICIT DIFFERENTIATION**

The Chain Rule is an important tool for developing what is called implicit differentiation. Most of our work thus far has been with explicit functions wherein one variable is explicitly presented as a function of another variable, such as our old quadratic equation, $y = ax^2 + bx + c$. There are many equations, such as $x^2 + y^2 = 4$, where it is implied that one variable is a function or several functions of the other variable. For example, the circle equation above can be solved for $y$ generating $y = \sqrt{4 - x^2}$ (which graphs as the upper semicircle) and $y = -\sqrt{4 - x^2}$ (which graphs as the lower semicircle). For many equations, it is either very difficult or impossible to solve for one variable in terms of another. Still more equations involve more than two variables. For these reasons, it would be nice to have another way to differentiate equations.

Consider the circle equation $x^2 + y^2 = 4$. Think of $f(x)$ as being one of the functions of $x$ that $y$ equals. Then $x^2 + f^2(x) = 4$. $\frac{d}{dx}[x^2 + f^2(x) = 4]$ produces $2x + 2f(x)f'(x) = 0$ and thus $f'(x) = -\frac{x}{f(x)}$. Now using Liebnetz's notation, we have $\frac{dy}{dx} = -\frac{x}{y}$. Observe that, unlike with explicit differentiation, the derivative is expressed in terms of both $x$ and $y$. Implicit differentiation is a very useful tool for dealing with more complex equations.

Now implicit differentiation can be used to extend the power rule to hold for fractional exponents. Consider $y = \sqrt{x} = x^{1/2}$. Squaring both sides produces $y^2 = x$. Differentiating implicitly yields $2y \frac{dy}{dx} = 1$. Thus $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2x^{1/2}} = \frac{1}{2} x^{-1/2}$.

**Exercise:** Use the same procedure as seen above to show that $\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{p/q - 1}$ where $p$ and $q$ are integers.

Although we will not prove it, the Power Theorem holds for any exponents. Because of the preceding development, we now can differentiate all algebraic functions and relations.
EXPONENTIAL AND LOG DERIVATIVES

Next up are exponential and log functions. If you need to, review these functions in chapter 7. The derivative of the general exponential function is developed below.

\[
\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}
\]

Now with some calculator work you can evaluate the limit seen above for various values for a. In particular, you should confirm the fact that if \( a = e \) (the irrational number approximated by 2.718, see previously), then the limit = 1. Thus:

\[
\frac{d}{dx} e^x = e^x.
\]

This remarkable fact that the function \( e^x \) is its own derivative again demonstrates that \( e \) is a special and useful number.

Now we will develop the log function's derivative. Consider \( y = \ln x \). In exponential form this is written as \( e^y = x \). Differentiating with respect to \( x \) yields \( e^y \frac{dy}{dx} = 1 \). Solving for \( \frac{dy}{dx} \) produces \( \frac{dy}{dx} = \frac{1}{y} \). Thus \( \frac{d}{dx} \ln x = \frac{1}{x} \).

Now consider a general log function \( y = \log_b x \). Using the change of base formula produces \( y = \frac{\ln x}{\ln b} \) and hence:

\[
\frac{d}{dx} \log_b x = \frac{1}{\ln b} \cdot \frac{1}{x}.
\]

Finally we return to the general exponential function \( y = a^x \). Taking the natural log of both sides produces \( \ln y = \ln a^x = x \ln a \). Differentiating with respect to \( x \) produces \( \frac{1}{y} \frac{dy}{dx} = \ln a \). Solving for \( \frac{dy}{dx} \) produces \( \frac{dy}{dx} = y \ln a = a^x \ln a \). Thus the rule for the derivative of \( a^x \) is:

\[
\frac{d}{dx} a^x = a^x \ln a
\]

In chain rule form the exponential and log derivative rules are:

\[
\frac{d}{dx} a^{f(x)} = a^{f(x)} \cdot \ln a \cdot f'(x) \quad \text{and} \quad \frac{d}{dx} \log_b f(x) = \frac{1}{\ln b} \cdot \frac{1}{f(x)} \cdot f'(x)
\]

TRIG DERIVATIVES

The next class of functions for which we will develop derivative rules are the trig functions.
Chapter 9: Introduction to Calculus

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
= \lim_{h \to 0} \left[ \sin x \left( \frac{\cos h - 1}{h} \right) + \cos x \frac{\sin h}{h} \right] = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}
\]

In order to evaluate the second limit (and subsequently the first one), some geometry and trig is required. Let P be a point on the unit circle and AT be tangent to the circle at A as shown below.

In terms of area, \( \text{Area}_{\Delta OAP} \leq \text{Area}_{\text{Sector OAP}} \leq \text{Area}_{\Delta OAT} \)

Thus \( \frac{1}{2} (1)^2 \sin \theta \leq \frac{\theta}{2\pi} \pi (1)^2 \leq \frac{1}{2} (1) \tan \theta \)

and \( \sin \theta \leq \theta \leq \tan \theta \)

and \( 1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta} \)

and \( 1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta \)

Now as \( \theta \to 0 \), \( \frac{\sin \theta}{\theta} \) will be "squeezed" to 1, and thus \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \).

**Exercise:** Now that you know that \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \), show that \( \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \).

So upon substituting \( \sin x \) in our expression for the derivative of \( \sin x \), we find that

\[
\frac{d}{dx} \sin x = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.
\]
To determine the derivative rule for \( \cos x \), recall that \( \cos x = \sin (\pi/2 - x) \), and use the chain rule.

\[
\frac{d}{dx} \cos x = \frac{d}{dx} \sin (\pi/2 - x) = \cos (\pi/2 - x) \cdot (-1) = - \sin x
\]

For \( \tan x \), use the quotient rule.

\[
\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]

Exercise: Derive the derivative rules for the other three trig functions.

**INVERSE TRIG DERIVATIVES**

Last up are the inverse trig functions which have their major utility in integration, as you will see in the next section.

First, consider \( y = \sin^{-1} x \) which is equivalent to \( \sin y = x \) for \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \).
Implicit differentiation yields \( \cos y \frac{dy}{dx} = 1 \)

Solving for \( \frac{dy}{dx} \) produces \( \frac{dy}{dx} = \frac{1}{\cos y} \)

Now for \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\), \( \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \). Thus:

\[
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}
\]

**Exercise:** Show that for \( y = \cos^{-1} x \), \( \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \)

Next, consider \( y = \tan^{-1} x \) which is equivalent to \( \tan y = x \) for \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\)

Implicit differentiation yields \( \sec^2 y \frac{dy}{dx} = 1 \)

Solving for \( \frac{dy}{dx} \) produces \( \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2} \). Thus:

\[
\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}
\]

**Exercise:** Show that for \( y = \cot^{-1} x \), \( \frac{dy}{dx} = -\frac{1}{1 + x^2} \)
Next, consider \( y = \sec^{-1} x \) which is equivalent to \( y = \cos^{-1} \frac{1}{x} \)

Differentiation yields \[
\frac{dy}{dx} = \frac{-1}{\sqrt{1 + \left( \frac{1}{x} \right)^2}} \cdot \left( \frac{-1}{x^2} \right) = \frac{\sqrt{x^2}}{x^2 \sqrt{1 + x^2}} = \frac{1}{x \sqrt{1 + x^2}}
\]

**Exercise:** Show that for \( y = \csc^{-1} x \), \[ \frac{dy}{dx} = -\frac{1}{|x| \sqrt{x^2 - 1}} \]

**INDETERMINATE FORMS**

In this section, you have seen the notion of a limit used to develop some differentiation formulas. In return, differentiation can be used as an aid in evaluating some types of limits that involve indeterminate forms such as \( \frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty, 1^\infty, 0^0, \text{ and } \infty^0 \). In 1696, the first calculus textbook was published. This book, written by the Frenchman Marquis de l'Hôpital (1661-1704), essentially was the lecture notes of his teacher, the Swiss mathematician, Johann Bernouli (1667-1748).
In this book appeared the misnamed *l'Hôpital's Rule* for evaluating
\[ \lim_{x \to a} \frac{f(x)}{g(x)} \] when both \( f(x) \) and \( g(x) \to 0 \) as \( x \to a \).

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

In other words, if both \( f \) and \( g \) approach 0, then first differentiate \( f \) and \( g \) separately, and then take the limit. As an example, consider the trig limit see previously.

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1
\]

This process can be repeated if necessary. This process can also be applied to limit fractions where both the numerator and the denominator approach infinity.

\[
\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{x^{1/3}}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{1}{\sqrt[3]{x}} = 0
\]

The next two indeterminate forms, \( \infty \cdot 0 \) and \( \infty - \infty \), can be modified so as to be able to apply *l'Hôpital's Rule*. Examples are shown below.

\[
\lim_{y \to \infty} y \sin \frac{1}{y} = \lim_{y \to \infty} \frac{\sin \frac{1}{y}}{\frac{1}{y}} = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1
\]

by letting \( x = \frac{1}{y} \) and observing that \( y \to \infty \Rightarrow x \to 0 \)

\[
\lim_{x \to 0^+} \left( x - \frac{1}{\sqrt[3]{x}} \right) = \lim_{x \to 0^+} \frac{1}{x^{1/3}} = \lim_{x \to 0^+} \frac{-1/2x^{-1/3}}{1} = -\infty \quad \text{DNE}
\]

The last three indeterminate forms, \( 1^\infty, 0^0, \) and \( \infty^0 \), also can be modified so as to be able to apply *l'Hôpital's Rule*. As an example, a familiar limit is evaluated below.

Let \( n = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x \Rightarrow \ln n = \lim_{x \to \infty} \ln \left( 1 + \frac{1}{x} \right)^x = \lim_{x \to \infty} \ln \left( 1 + \frac{1}{x} \right)^x = \lim_{x \to \infty} x \ln \left( 1 + \frac{1}{x} \right) \)

\[
= \lim_{x \to \infty} \frac{\ln \left( 1 + \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{1}{x + 1} \left( \frac{1}{x} \right)^x}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{(1 + \frac{1}{x})} = 1
\]

Thus \( \ln n = 1 \) and hence \( n = e \).
Problem Set 9.3

Find \( \frac{dy}{dx} \) and simplify. [ 1 - 4 ]

1. \( y = 3x^4 \cdot \frac{3}{x^3} + \sqrt{x} \)
2. \( y = \frac{\tan(5x^2 - 2)}{2x + 3} \)
3. \( y = \sec^5(3x - 2) \)
4. \( y = (3x - 2)^2 \sqrt{4x + 3} \)

Evaluate the following limits [5 - 6].

5. \( \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{\cos 2x} \)

6. \( \lim_{\theta \to 0} \frac{A(\theta)}{B(\theta)} \), For this problem, \( A(\theta) \) represents the area of a semicircle surmounted on an isosceles triangle with vertex angle \( \theta \) and area \( B(\theta) \).

7. Let \( f(0) = 1, g(0) = -2, h(0) = 3, f'(0) = -4, g'(0) = 5, h'(0) = -6 \) and \( y = f(x) \cdot g(x) \cdot h(x) \). Find \( y'(0) \).

8. Write the equation of the normal line to the graph of \( y = \sin 2x \) at the point where \( x = \pi/3 \).

9. Ball Problem: A bottle rocket is shot vertically upward from a street in such a manner that it falls on a roof 80 feet above the street. Its distance, \( s \) (in feet), from the street is given by \( s = 96t - 16t^2 \) where \( t \) is seconds. Find (a) the maximum height above the street the rocket reaches and (b) its velocity when it strikes the roof.

10. Find \( (f^{-1})'(2) \) for \( f(x) = x^2 - x^3 + 2x \).
11. Determine where the normal to the ellipse, \( x^2 - xy + y^2 = 3 \) at \((-1,1)\) intersects the ellipse a second time.

Find \( \frac{dy}{dx} \) and simplify. \([12 - 13]\)

12. \( y = (\ln x)^\cos x \)

13. \( y^5 + x^2y^3 = 1 + x^4y \)

14. Find and simplify \( f'''(1) \) for \( f(x) = e^{2x}\ln 2x \).

15. Find an equation for the tangent line to the curve \( 2e^{xy} = x + y \) at the point \((0,2)\).

16. Determine \( D(35) (x\sin x) \).

17. A particle is moving along the x-axis with its position \( x \) as a function of time \( t \) given by \( x = t^3 - 12t^2 + 36t \). Calculate the velocity and acceleration functions and then find when is the particle accelerating to the right.

18. Use implicit differentiation to show that a tangent to a circle is perpendicular to the radius drawn to the point of tangency. \([\text{Use } x^2 + y^2 = r^2 \text{ for the equation of the circle.}]\)

19. Calculate \( \lim_{x \to 0} \frac{\tan x - x}{x^3} \)

20. Calculate \( \lim_{x \to 0,5} (1 + \sin 4x)^{\cot x} \)
9.4 APPLICATIONS OF THE DERIVATIVE

We now know how to differentiate all of the elementary functions, and we possess the powerful tools of the power rule, chain rule and implicit differentiation. So what do we use this for? In this section we will look at related rates, differential equations, curve sketching, optimization problems and other ideas associated with the derivative.

RELATED RATES

Recall that a derivative can be thought of as a rate at which one quantity changes with respect to change in another quantity. For example, the rate at which the area of an expanding circle changes is directly related to the rate at which the radius changes. Related rate problems, as the name suggests, involve relating rates.

Example: Suppose the radius of a circle is currently 3 cm and is increasing at the rate of 2 cm/sec. At what rate is the area of the circle changing at the instant when \( r = 5 \text{ cm} \)?

Solution: First, relate the variables \( A = \pi r^2 \).

Second, differentiate with respect to time \( t \)

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

Finally, substitute 5 cm for \( r \) and 2 cm/sec for \( \frac{dr}{dt} \) to find;

\[
\frac{dA}{dt} (r = 5\text{ cm}) = 2 \pi (5 \text{ cm})(2 \text{ cm/sec}) = 20 \pi \text{ cm}^2 /\text{sec}
\]

Even though some problems are more complicated than others (as you will see in the problem set for this section), all related rate problems are solved in essentially the same manner.

DIFFERENTIAL EQUATIONS

Generally, related rate problems, such as seen in the example above, involve rates of the first order. A first derivative is called a first order rate. A second derivative is called a second order rate, etc... Previously, you have worked with calculating rates of various order. In the next subsection you will be using first and second derivatives as a tool to aid in curve sketching. Later in this chapter you will solve differential equations which are equations that involve relationships between functions and their derivatives of various order. For example, recall back in Chapter 7, when examining exponential growth, we discussed how the growth/decay formula \( A = A_0 e^{kt} \), is the solution to the rate equation \( \frac{dA}{dt} = kA \). This differential equation is called an ordinary, first order, first degree D.E.. It is ordinary as opposed to partial, which involve derivatives produced by differentiating equations such as \( z = f(x,y) \) where there are multiple independent
variables. It is first order because the highest derivative it involves is a first derivative. With regard to degree, recall that for a polynomial, its degree is the sum of the exponents of all the variables in the highest degree term. For example, the polynomial \( x^2 + 2xy^2 + 3x^2yz^3 \), has degree 6. For a DE, its degree is the sum of the exponents of all the variables and derivatives in the highest order term. For example the DE, \( x + y(y')^2 + xy'' = 0 \) is second degree. Our growth/decay DE is first degree.

In physics, a major application of calculus occurs in solving Ballistics problems, which involve differential equations. Let an equation of the position of some object in motion as a function of time be given by \( s = f(t) \). Then over a time interval \((t_1, t_2)\), the object's position changes by an amount \( f(t_2) - f(t_1) \). Dividing the position change by the time change produces
\[
\frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]
which is called the average velocity over the time interval. Now as the time interval decreases to 0, the position change does also, and as you saw at the end of section 9.2, this limit is a derivative which in Ballistics is called the instantaneous velocity, or just velocity, and is given by \( v = \frac{ds}{dt} = f'(t) \). In turn, the instantaneous rate of change of velocity as a function of time, which is called acceleration, is given by \( a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t) \). A typical ballistics problem for which the position equation is known is shown below. The solution, which involves concepts presented in the next two subsections, is also shown (so as to provide motivation for those next sections).

**Problem:**

An object is projected upward from the surface of the Earth, at an initial velocity of 160 ft/sec. It reaches an altitude of \( s = 160t - 16t^2 \) ft in \( t \) sec.

(a) How high does it reach?
(b) With what velocity does it strike the ground?

**Solution:**

The object reaches its maximum altitude when its velocity drops to zero. Thus, since \( v = 160 - 32t \), when \( t = \frac{160}{32} = 5 \) sec them maximum altitude of \( s(5) = 400 \) ft is reached.

Impact velocity occurs when the object strikes the ground, which is when \( s = 0 \). \( s = 160t - 16t^2 = 16t(10 - t) = 0 \) when \( t = 0 \) and again when \( t = 10 \) sec. \( V(10) = 160 - 32\cdot 10 = -160 \) ft/sec. Observe that \( v \) is negative as it should be, because the object is plunging towards the Earth.

Perhaps the question, "Where did this equation for \( s \) come from?", may have occurred to you. Just as did the growth/decay equation, it comes as a result of solving a
differential equation, as you will later in this chapter. Finding solutions to differential equations can be a difficult and tedious process, especially if done by hand. Fortunately, high speed computers can be programmed to solve these messy DE’s. It has been said that this country's space program could not have gotten off the ground (pun intended) without the use of computers to solve the complex DE’s fast enough to provide timely instructions for ballistics concerns, such as burn time, thrust vectors, etc.. The study of differential equations provides enough material for an entire mathematics course (typically after three semesters of calculus). However, except for a little more discussion later in this chapter, we will not spend any time with differential equations.

CURVE SKETCHING

Previously we have developed several techniques which aid the production of a nice sketch of some function or relation. We know how to find intercepts and asymptotes, identify domain and range, recognize various types of symmetry, and use transformations on basic library or catalogue graphs to produce graphs of a wide collection of functions and relations. Derivatives provide the last tools we need for graphing.

First of all, recall that the derivative is a formula for the slope of a curve. It's no stretch to realize that if a derivative at a point is positive, then the slope at that point is positive and thus the curve is rising to the right. Likewise, if a derivative at a point is negative, the slope at that point is negative, and thus the curve is falling to the right. Finally, if a derivative at a point is zero, the slope at that point is zero, and thus the curve has a horizontal tangent at that point. Such a point is called a stationary point.

Now suppose a curve is stationary at some point A, rising left of A and falling right of A as shown on the left below.

Clearly it is reasonable to say A is a maximum point on this curve. Likewise, if a curve is stationary at some point B, falling left of B and rising right of B as shown on the right above, then B is a minimum point on the curve.

Now examine the curve shown below.

Stationary Points A and C are each maximum points relative to nearby points and Stationary Points B and D are each minimum points relative to nearby points. Such points are called relative or local maximum or minimum points respectively. Now on the interval from E to F, Point A is the absolute or global maximum point on the graph, and
likewise, point D is the absolute or global minimum point on the graph. Now, if we restrict ourselves to just the interval from E to F, then E is a relative minimum point, and, likewise, F is a relative maximum point. Now points A, B, C and D are referred to as being **turning points** of the graph. Turning points are often the most important points on a graph as we will see in the next subsection. The turning points at A, B, C, and D each occur as stationary points where the slope of the tangent line is zero. These points occur in smooth transitions of the curve from increasing to decreasing or vice versa. Curves which have smooth transitions are called **smooth curves**. All polynomial graphs are smooth curves.

**ROLLE'S THEOREM**

By now you should know that maximums and minimums can occur at points where the derivative equals 0. **Rolle's Theorem** is a useful result related to this concept.

Suppose a function, $f(x)$, is continuous on $[a, b]$ and differentiable on $(a, b)$. Further, suppose $f(a) = f(b) = 0$.

Then there exists at least one number $c \in (a, b)$ such that $f'(c) = 0$.

You have actually seen this result before. Recall that the vertex of a parabola with real zeros is "halfway" between the zeros. In general maximum and minimum points are not "halfway" between real zeros, but Rolle's Theorem guarantees the existence of these turning points between successive real zeros of continuous differentiable functions. The concept embedded in Rolle's Theorem is pictured below.

Another way to think about Rolle's Theorem is to first realize that the slope of the secant line through $(a,0)$ and $(b,0)$ is 0. Second, the slope of the tangent line(s) to the curve at a point guaranteed by Rolle's Theorem is also 0. Thus we can say that Rolle's Theorem guarantees at least one point $(c, f(c))$ where $c \in (a, b)$ such that the tangent line at the point is parallel to the secant line through $(a,0)$ and $(b,0)$.

**THE MEAN VALUE THEOREM**

A generalization of Rolle's Theorem, which is called the **Mean Value Theorem** (MVT), is stated below.
If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that the slope of the tangent line at \((c, f(c))\) equals the slope of the secant line through \((a, f(a))\) and \((b, f(b))\). As an equation, this result is stated as \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

A picture is shown below. An analytic proof of the MVT, which is based on Rolle's Theorem and is quite nice, follows the picture.

Consider moving secant line \( AB \) vertically until it becomes a tangent line (at \( P \)).

\( P \) will occur at the point where the vertical distance between the secant line and the tangent line is a maximum.

\[
\frac{RQ}{AQ} = \frac{BD}{AD} = \frac{f(b) - f(a)}{b - a} = m \text{ sec} \Rightarrow RQ = m \cdot AQ = m(x - a)
\]

\[ RP = RQ + QP = m(x - a) + f(a) \]

\[ SR = SP - RP = [f(x) - 0] - [m(x - a) + f(a)] \]

Thus \( SR \) is a function of \( x \), say \( H(x) = f(x) - m(x - a) - f(a) \)

Observe that \( H(a) = H(b) = 0 \), so by Rolle's Theorem, \( \exists \ c \in (a, b) \ \ni \ H'(c) = 0 \)

So, since \( H'(x) = f'(x) - m, H'(c) = f'(c) - m = 0 \) \( f'(c) = m = \frac{f(b) - f(a)}{b - a} \) \( Q.E.D. \)

**Note:** Q.E.D. is an abbreviation for the Latin phrase *quod erat demonstrandum* which means that which was to be proved, and is often seen at the conclusion of proofs. Some of us believe that Q.E.D. stands for *Quickly Evade Discussion*.

You will see the MVT utilized in the next section. In fact, because the MVT is so elegant and useful, some of us believe it should be designated as the *Fundamental Theorem of Differential Calculus*. But for now, we return to curve sketching.

**THE FIRST DERIVATIVE TEST**

You have seen that for smooth curves, a maximum occurs at a stationary point \( P \), where the curve is rising left of \( P \) and falling right of \( P \). Likewise a minimum occurs at a
stationary point $P$, where the curve is falling left of $P$ and rising right of $P$. The result is typically called the **First Derivative Test**.

**THE SECOND DERIVATIVE TEST**

Now \( \frac{d}{dx} [y = f(x)] \) yields \( \frac{dy}{dx} = f'(x) \). \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \) yields what is called the second derivative. Algebraically \( \frac{d}{dx} \left( \frac{dy}{dx} \right) \) would equal \( \frac{d^2y}{(dx)^2} \). Over time, the parentheses were omitted and today, we indicate the second derivative of $y$ with respect to $x$ twice as $\frac{d^2y}{dx^2}$. In terms of primes, the second derivative is indicated by $y''$ or $f''(x)$.

The first derivative gives the slope of a curve, which in terms of $x$ and $y$, is the instantaneous rate of change in $y$ with respect to change in $x$. Likewise, the second derivative would be the instantaneous rate of change in $y'$ with respect to change in $x$ symbolized as $\frac{d}{dx} y'$. Shown below are the graphs of a polynomial function, its first derivative and its second derivative.

![Graphs of a polynomial function, its first derivative and its second derivative.](image)

Observe that, as we have discussed previously, when $y' > 0$, $y$ is increasing; when $y' = 0$, $y$ is stationary; and when $y' < 0$, $y$ is decreasing. Likewise when $y'' > 0$, $y'$ is increasing; when $y'' = 0$, $y'$ is stationary; and when $y'' < 0$, $y'$ is decreasing. Now with regard to the relationship between $y''$ and $y$, first observe that when $y'' > 0$, $y'$ is increasing. The graph of $y$ is bending up, which is called **concave up**. Second, observe that when $y'' < 0$, $y'$ is decreasing. The graph of $y$ is bending down, which is called **concave down**. Finally,
when \( y'' = 0 \), \( y' \) is stationary, and \( y \) is changing from concave up to down or down to up. A point where concavity changes is called an **inflection point**.

The preceding observations lead to what is called the **second derivative test**.

- If \( y'' > 0 \) the curve is concave up.
- If \( y'' < 0 \) the curve is concave down.
- If \( y'' = 0 \) the curve is at an inflection point.

The first and second derivative tests used in conjunction with each other provide a straightforward procedure for finding maximums and minimums.

- If \( y'(a) = 0 \) and \( y''(a) > 0 \), then \((a, f(a))\) is a minimum point.
- If \( y'(a) = 0 \) and \( y''(a) < 0 \), then \((a, f(a))\) is a maximum point.
- If \( y''(a) = 0 \), then \((a, f(a))\) is an inflection point.*

*Note:* This is true provided \( a \) is a root of odd multiplicity of \( y'' = 0 \). For example consider \( y = x^4 \). \( y' = 4x^3 \) and \( y'' = 12x^2 \) which is 0 when \( x = 0 \), but \((0,0)\) is a minimum point on the graph.

We have been discussing what occurs when \( y' \) and \( y'' \) equal 0. Maximum, minimum and inflection points also occur at points where \( y' \) and \( y'' \) are undefined. Consider \( y = x^{2/3} \) which is graphed below.

![Graph](image1)

\[
y' = \frac{2}{3} x^{-2/3} \Rightarrow y'(0) \text{ DNE} \quad y'' = -\frac{2}{9} x^{-4/3} \Rightarrow y''(0) \text{ DNE}
\]

Observe the tangent to the curve at \( x = 0 \) is a vertical line for which the slope DNE. Also observe that \((0,0)\) is a minimum point and that \( y \) is always concave down.

Next consider \( y = 4 - x^{2/3} \) which is graphed below.

![Graph](image2)

Again \( y' \) DNE at \( x = 0 \); \( y'' \) DNE at \( x = 0 \). For this function \((0,4)\) is a maximum point and the curve is always concave up.

Finally consider \( y = \sqrt[3]{x} \) which is graphed below.
Chapter 9: Introduction to Calculus

\[ y' = \frac{1}{3} x^{-2/3} \quad y'' = -\frac{2}{9} x^{-5/3} \]

- \( y' \) DNE at \( x = 0 \) \( \Rightarrow \) a vertical tangent line at \( (0,0) \)
- \( y'' < 0 \) for \( x > 0 \) \( \Rightarrow \) concave down for \( x > 0 \)
- \( y'' > 0 \) for \( x < 0 \) \( \Rightarrow \) concave up for \( x < 0 \)
- \( y'' \) DNE at \( x = 0 \) and \( (0,0) \) is an inflection point

To summarize what occurs when \( y' \) and or \( y'' \) DNE at a point \( (a, f(a)) \) we note that:
- \( y'(a) \) DNE \( \Rightarrow \) a vertical tangent line at \( (a, f(a)) \). The point \( (a, f(a)) \) may or may not be a maximum of minimum point.
- \( y''(a)\) DNE \( \Rightarrow \) The point \( (a, f(a)) \) may or may not be an inflection point.

The chart shown below provides a **Summary of Curve Sketching**.

1. Determine the domain. Check for points of discontinuity.
2. Find and plot the intercepts.
3. Find equations for and graph the asymptotes, if any.
4. Identify any symmetries that exist.
5. Determine, using the first derivative, intervals where the graph is increasing or decreasing
6. Determine, using the first and second derivatives, and plot local maximum and minimum points.
7. Determine intervals for up and for down concavity using the second derivative.
8. Determine, using the second derivative, and plot inflection points.
9. Sketch the graph.

For many applications of curve sketching, finding maximum and minimum points is of primary importance. In general a point \((x, f(x))\) is a **candidate** to be a maximum or minimum point if it is (a) an endpoint of an interval of interest, (b) a corner or cusp point, (c) a point where \( f'(x) \) is either 0 or undefined. Such points are often referred to as being **critical points** and the abscissa of the point is called a **critical number**. This important endeavor of finding critical numbers and then determining if they determine maximum or
minimum values for the corresponding ordinate variable is discussed in the next subsection.

**OPTIMIZATION**

Consider the following problem which was both real and personal.

Scotty's Auto Sales specializes in inexpensive small and mid-size autos. This little company has been selling an average of 10 cars per month at an average price per car of $4000. Recent experience has shown that for each $100 decrease (increase) in the average selling price, they averaged 2 more (less) sales per month. What selling price should Scotty set so as to maximize the sales revenue?

This is an example of an optimization problem, wherein one wishes to optimize one variable for changes in another variable. In the problem above, we desire to maximize revenue for all possible choices for the selling price. So if we express the revenue \( r \), as a function of the selling price \( s \), we can find the value for \( s \) that produces a maximum value for \( r \).

The expression \( \frac{4000 - s}{100} \) would represent the number of $100 decreases in the price. Hence the average number of sales per month would be given by \( 10 + 2 \times \frac{4000 - s}{100} \).

Thus the revenue equation is \( r = s \times \left[ 10 + \frac{4000 - s}{50} \right] \).

At this point we have a choice as to what to do next. We can just take the derivative, using the product rule, or we can simplify the equation first, and then take the derivative. Experience, personal preference and technology available dictates which way you go. We are going to simplify first and then set the derivative equal to 0 to find an optimal point.

\[
\frac{dr}{ds} = 90 - \frac{1}{25}s = 0 \Rightarrow s = 2250
\]

\[
\frac{d^2r}{ds^2} = - \frac{1}{25} < 0 \Rightarrow s = 2250 \text{ maximizes } r.
\]

Thus Scotty's Auto Sales should price their cars at an average of $2250 each.

Perhaps you noticed that finding the equation was the difficult part of the solution. What I think is an easier approach to this problem is to begin by letting \( x \) equal the number of $100 decreases in the selling price. Then the selling price is given by \( 4000 - 100x \) and the number of sales is given by \( 10 + 2x \).

\[
r = (4000 - 100x) \times (10 + 2x) = -200x^2 + 7000x + 40000
\]

\[
\frac{dr}{dx} = -400x + 7000 = 0 \Rightarrow x = 17.5
\]

\[
\frac{d^2r}{ds^2} = -400 < 0 \Rightarrow x = 17.5 \text{ maximizes } r.
\]

The sales price should be \( 4000 - 17.5 \times 100 = $2250 \)
Of course Scotty's Auto Sales was not interested in maximizing revenue. The goal was to maximize profit. Suppose the cost per car averaged $3000. As an exercise, in the space below, determine the average selling price that would maximize profit.

**Exercise space:**

Optimization is an important mathematical process applied to many different problem situations in many different walks of life. We will conclude this section with one more example, and then you can do more problems in the problem set that follows.

Your mission is to design an oil pipeline to run from an drilling platform located 5 km offshore to a refinery located down shore 13 km from the point directly ashore from the platform. If it costs 1.4 times as much to lay pipe along the sea bed as to lay it in a trench on land, where should you bring the pipe ashore so as to minimize cost? A picture depicting the problem setting is shown below.

![Diagram of the pipeline problem](image)

Let $k =$ cost per km to lay pipe on land. Then:

$$C = 1.4k\sqrt{25 + x^2} + k(13 - x) \quad 0 \leq x \leq 13$$

$$\frac{dC}{dx} = \frac{1.4kx}{\sqrt{25 + x^2}} - k = 0 \Rightarrow 1.4kx = k\sqrt{25 + x^2}$$

$$\Rightarrow 1.4x = \sqrt{25 + x^2} \Rightarrow 1.96x^2 = 25 + x^2 \Rightarrow 0.96x^2 = 25$$

$$\Rightarrow x = \frac{5}{\sqrt{0.96}} \approx 5.1$$

Using the second derivative will confirm that this value for $x$ minimizes the cost.

Thus laying the pipe underwater to a point 5.1 km down the shore towards the refinery will produce minimum cost.

**Problem Set 9.4**

1. **Graphing Problem:** Sketch the graph of $y = x^3 - 9x^2 - 21x - 11$ showing exact coordinates of the max, min and inflection points.
9.4 : Applications of the Derivative

2. **Graphing Problem II:** Graph showing exact intercepts, asymptotes, max, min, and inflection points. 

   \[ y = \frac{x^2}{1 - x^2} \]

3. **Lake Problem:** Lake Clemson is created to be in the shape of an inverted cone with a diameter of 48 meters, and a depth of 10 meters. When \( L = 13 \) meters, it is observed to be increasing at the rate of 50 cm per day. How much longer will it take the lake to fill, assuming a constant inflow of water?

4. **Light Problem:** A woman 6 feet tall is running at 15 ft/sec away from a street light which is 15 feet above the ground. At what rate is the tip of her shadow moving?

5. **Lighthouse Problem:** A light in a lighthouse located 1 km offshore from a straight shoreline is revolving at 2 rpm. How fast is the light beam moving along the shore when it is 1/2 km down from the point directly onshore from the lighthouse?

6. **Train Problem:** At noon an eastbound train, which is 25 miles due north of a northbound car, is traveling at 40 mph. If the car is traveling at 50 mph, (a) find the rate at which the distance between the car and the train is changing at 12:12 p.m. and (b) find the time when the two are closest together.

7. **Sector Problem:** If the perimeter of a circular sector is 100 feet, what values of radius and arc length will give the sector the greatest area?

8. **Vat Problem:** An open-top rectangular stainless-steel vat is to have a square base and a volume of 32 cu. ft. If it is to be welded from quarter-inch plate, what dimensions will produce the vat with minimum weight?

9. **Oil Rig Problem:** A drilling rig is 5 miles offshore. A pumping station is 10 miles down the shore from the point directly onshore from the rig. A man is in a boat at the rig. If he can travel 15 mph on water and 39 mph on land, at what point should he land the boat so as to minimize the time required to get to the pumping station?

10. **Dealer Problem:** Scotty's Auto Sales sells an average of 10 cars per week at an average price per car of $4000. Experience has shown that for each $100 increase (decrease) in price, two less (more) cars are sold per week. What selling price will maximize profits if the average car costs the dealer $3000?

11. **Warehouse Problem:** A rectangular warehouse that is to have two rectangular rooms separated by one interior wall must have 5000 square feet of floor space. The cost of exterior walls is $150 per linear foot, and the cost of interior walls is $90 per linear foot. Find the dimensions of the least expensive warehouse.

12. **Barnyard Problem:** The Farmer Fairbairns have 220 feet of fencing with which to enclose a barnyard having perpendicular sides using 220 feet of fencing along with
any portion of the sides of their 60 foot by 40 foot barn. The barn could be completely inside the barnyard, in one corner of the barnyard, or along part or all of one side of the barnyard as shown below (or perhaps located in some other manner).

(a) Determine the area of the barnyard as a function of the width, w, for various possible configurations.
(b) Find the maximum possible area for the barnyard.

13. **Sidewalk Problem:** After many years of remodeling the barn and having the city grow up around the farm, the old farmers sold their farm to the city. The old barn is now a 27' by 64' building which houses a country music museum and is located on an intersection of two streets, as shown in the diagram below. The city wants to build a straight-line sidewalk from one street to the other, so that the sidewalk just touches the corner of the building. Show that the angle $\theta$, which minimizes the distance $d$ along the walk, is given by $\tan \theta = 4/3$.

14. **Plane Problem:** After selling their farm the old farmers take a trip to Europe. At a certain instant, their airplane A is flying a level course at 500 mi/hr. At the same time, airplane B is straight above airplane A, flying at 700 mi/hr on a slant course that intercepts A's course at a point C that is 4 miles from B and 2 miles from A.

(a) At the instant in question, how fast is the distance between the airplanes decreasing?
(b) To the nearest whole foot, what is the minimum distance between the airplanes, if they continue on the present courses at constant speed?
(c) Do the farmers survive this exciting adventure?
9.5 THE INTEGRAL

Most of calculus as we know it today, was developed in the Eighteenth and Nineteenth centuries, chiefly by Newton and Leibnitz. As mentioned in section 9.1, the beginnings of integral calculus originated with the Greeks over two millennium ago as they attempted to find areas of non-polygonal regions. Newton and Leibnitz realized that anti derivatives could be used to find areas. The process of finding a function when its derivative is known is an integral part (pun intended) of finding area. The notation we use for integration actually comes from the limit of the sum of areas of rectangles process for finding area that we saw in section 9.1. Consider finding the area from 0 to 2 under the graph of \( y = \frac{x^2}{5} + 1 \).

If we partition the interval \([0,2]\) into \(n\) subintervals of equal width then each width is \(\frac{2}{n}\). The area is the limit, as the number of rectangles \(n \to \infty\), of the common width times the sum of the heights of the \(n\) rectangles.

\[
\begin{align*}
A_0^2 &= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left[ \left( \frac{i \cdot \frac{2}{n}}{5} \right)^2 + 1 \right] = \frac{38}{15}
\end{align*}
\]

This partitioning is a uniform partitioning with each strip (and the approximating rectangle) having the same width. Additionally, we calculated the heights of the rectangles by substituting the \(x\) coordinate of the right end of each subinterval into the function. These two restrictions are not necessary in order to calculate the area of the region. Any partitioning will do, and any \(x\)-value in each interval can be used to calculate heights. If we let \( \Delta x_i \) represent the width of the \(i^{th}\) strip (and rectangle) under the graph from \(a\) to \(b\) of some positive-valued function \(f(x)\), and \(c_i\) be any \(x\) value in the \(i^{th}\) interval, then the area is given by:

\[
A_a^b = \lim_{\| \Delta x \| \to 0} \sum_{i=1}^{n} [f(c_i)] \Delta x_i \quad \text{where} \quad \| \Delta x \| \text{ denotes the width of the largest subinterval.}
\]

Observed that if the width of the largest subinterval goes to 0, then all subinterval widths go to 0. Also realize that as the interval widths approach 0, then \(n \to \infty\). The expression seen above is known as a Reimann Sum, so named in honor of the German mathematician, Georg Friedreich Bernhard Reimann (1826-1866).

THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS
Now, mathematicians had for some time suspected that there was a connection between the Riemann sum expression for an area and an anti derivative. The expression of the relationship forms the **Fundamental Theorem of Integral Calculus (FTIC)**.

Consider the region from a to b under the graph of a positive-valued continuous function \( y = f(x) \) which is partitioned into n strips as shown below.

Let \( \Delta x_i \) equal the width of the \( i^{th} \) interval and let \( F(x) \) be a function such that \( F'(x) = f(x) \).

By the **Fundamental Theorem of Differential Calculus** (the MVT), there exists a \( c_i \) in each interval such that for the function \( F(x) \):

\[
\begin{align*}
F(a + \Delta x_1) & - F(a) = F'(c_1) \Delta x_1 = f(c_1) \Delta x_1 \\
F(a + \Delta x_1 + \Delta x_2) & - F(a + \Delta x_1) = F'(c_2) \Delta x_2 = f(c_2) \Delta x_2 \\
F(a + \Delta x_1 + \Delta x_2 + \Delta x_3) & - F(a + \Delta x_1 + \Delta x_2) = F'(c_3) \Delta x_3 = f(c_3) \Delta x_3 \\
& \vdots \\
F(b) - F(a + \Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_{n-1}) & = F'(c_n) \Delta x_n = f(c_n) \Delta x_n 
\end{align*}
\]

Summing the equations produces a telescoping series on the left and thus:

\[
F(b) - F(a) = \sum_{i=1}^{n} f(c_i) \Delta x_i.
\]

Now as the width of each interval \( \to 0 \), the number of intervals \( n \to \infty \).

Thus:

\[
F(b) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i.
\]

This says that the area calculated by Riemann sums is equal to the anti derivative evaluated at the right side of the region minus the anti derivative evaluated at the left side of the region.

In terms of integral notation, we indicate this area as:

\[
\int_{a}^{b} f(x) \, dx.
\]

The integral symbol is an elongated S and represents sum. The limits of integration (a to b) represent finding the area from a to b. \( c_i \) is a number in the interval \( (x, x + \Delta x_i) \) so as \( \Delta x_i \to 0 \), \( c_i \to x \). Hence you can think of \( f(x) \, dx \) as the product of the height and width of a representative rectangle. So our integral says to sum up all the number, \( n \to \infty \), of rectangles with area approaching 0 between a and b. **So we could say an infinite**
number of nothings add up to something - indeed a perplexing yet powerful notation.

At the beginning of this section we found the area from 0 to 2 under the graph of \( y = \frac{x^2}{5} + 1 \) to equal \( \frac{38}{15} \). By integration, the area is calculated to be:

\[
\int_{0}^{2} \left( \frac{x^2}{5} + 1 \right) dx = \frac{x^3}{15} + x \bigg|_{0}^{2} = \frac{8}{15} + 2 = \frac{38}{15}
\]

(which is the same as before).

THE MEAN VALUE THEOREM FOR AREA

You saw the MVT in action when proving the FTIC. Thinking of the entire interval, observe that, by the MVT, there would exist some number \( c \) in \((a, b)\) such that \( F(b) - F(a) = f(c) \cdot (b - a) \). This means that there exists a rectangle with the same base as the interval and having the same area as under the curve. In other words, the area under the curve is the mean (not the average) area between the area of the single inscribed rectangle and the single circumscribed rectangle. This concept is best explained with a picture.

As the title of this subsection indicates, this result is often called the **MVT for Area**. In addition to seeing this notion used in the next subsection, you will also see it used in an application of integration in the next section.

THE FUNDAMENTAL THEOREM OF CALCULUS

Now, the relationship between integration and Riemann sums in finding area suggests that an integral can be used to define a function. The integral from \( t = a \) to \( t = x \) of any continuous function \( f(t) \) defines a number \( g(x) = \int_{a}^{x} f(t) \, dt \), which can (in theory) be computed. Now, \( g(x) \) depends only on \( x \) and as \( x \) changes, so does \( g(x) \). Observe that:

\[
g(x + \Delta x) - g(x) = \int_{a}^{x+\Delta x} f(t) \, dt - \int_{a}^{x} f(t) \, dt = \int_{x}^{x+\Delta x} f(t) \, dt
\]
This last integral is the area of the strip from \( x \) to \( x + \Delta x \), which by the MVT for Area, 
\[ g(x + \Delta x) - g(x) = f(c)\Delta x \]
and 
\[ \frac{g(x + \Delta x) - g(x)}{\Delta x} = f(c) \]
And as \( \Delta x \to 0 \) we have 
\[ g'(x) = f(x) \]
Thus: 
\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

Roughly speaking, the **Fundamental Theorem of Calculus** says that the derivative of an integral produces the original function. In other words, differentiation and integration are inverse operations.

**Example:** 
\[ \frac{d}{dx} \int_0^x 3t^2 \, dt = 3x^2 \]

The fundamental theorem can be extended to handle the case where the upper limit is some function of \( x \) as shown below.

Let 
\[ g(x) = \int_{\sin x}^{\sin x} 3t^2 \, dt = t^3 \int_{\sin x}^{\sin x} = \sin^3 x - 8 \]

Then 
\[ g'(x) = 3 \sin^2 x \cdot \cos x \]

Generalizing this concept, 
\[ \frac{d}{dx} \int_a^{h(x)} f(t) \, dt = f[h(x)]h'(x). \]

This result is similar to the chain rule seen in derivatives.

**Exercise:** Use the area concept that if \( a < b < c \), then \( A_a^c = A_a^b + A_b^c \) to explain why 
\[ \frac{d}{dx} \int_a^{h(x)} f(t) \, dt = f[h(x)]h'(x) - f[k(x)]k'(x). \]

**Hint:** Let some number \( n \) be between \( k(x) \) and \( h(x) \).

Now anti derivatives are sometimes difficult, and in fact, sometimes impossible, to find. Hence, before continuing to study the process of finding areas, and subsequently, other applications of integration (in the next section), we will examine some methods of finding anti derivatives. Anti derivatives are more commonly referred to as indefinite* integrals and indicated by: 
\[ \int f(x) \, dx = F(x) + C \]

\( C \) is added so as to provide the most general anti derivative. 
\[ \frac{d}{dx} [f(x) + C] = f(x). \]

**Note:** As distinguished from definite integrals, which have limits and calculate some definite quantity.

**THE POWER RULE FOR INTEGRALS**
First of all, knowing derivatives of some functions allows us to think backward and write some integral formulas. \( \frac{d}{dx} x^n = nx^{n-1} \) This says in order to differentiate \( x^n \), decrease the exponent by 1 and multiply by the old exponent.

Thus to integrate \( x^n \), you increase the exponent by 1 and divide by the new exponent. This rule, commonly called the Power Rule for Integration, is shown below.

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1
\]

Note: We will see a rule for the case when \( n = -1 \) later in this section.

Next, since we know the derivative of a sum is the sum of the derivatives, the same is true in integration. The integral of a sum is the sum of the integrals. Consequently, we can easily integrate all polynomials; and some rational functions (namely the ones that can be expressed as a sum of terms of the form \( a_n x^n, n \neq -1 \)).

SUBSTITUTION

Consider \( \int \frac{x}{(x^2 + 3)} \, dx \). If we perform a substitution, as follows, we can evaluate this integral.

Let \( u = x^2 + 3 \). Then \( \frac{du}{dx} = 2x \), and \( x \, dx = \frac{1}{2} \, du \).

Substituting in the integral produces:

\[
\int \frac{1}{2} \frac{du}{u^4} = \frac{1}{2} \int u^{-4} \, du = \frac{u^{-3}}{-6} + c = -\frac{1}{6u^3} + c = -\frac{1}{6(x^2 + 3)^3} + c
\]

Now consider \( \int x\sqrt{3x^2 - 5} \, dx \). If we let \( u = 3x^2 - 5 \), then \( du = 6x \, dx \), and we write the integral as \( \frac{1}{6} \int u^{1/2} \, du \) which equals \( \frac{1}{6} \frac{u^{3/2}}{3/2} + c = \frac{1}{9} (3x^2 - 5)^{3/2} + c \). Hence, we see that some algebraic functions can be integrated.

Next consider \( \int (x^2 + 3)^{100} \, dx \). If we try a substitution on the problem by letting \( u = x^2 + 3 \), then \( \frac{du}{dx} = 2x \), and again \( \frac{1}{2} \, du = x \, dx \). But this time we can't get \( du \) to go with \( u \) (You can't go to the dance if you don't get \( du \) to go with \( u \)). Thus you realize that substitution does not always work. Now the integrand in this example can be integrated, but you would have to either multiply out the integrand (ugh!) or do something else. Integration is more challenging than differentiation.

INTEGRATION BY PARTS
Shortly, we will develop integration rules for the basic exponential, log, trig and inverse
trig functions. To integrate \( \ln x \) requires the method of integration by parts. Recall the
product rule \( \frac{d}{dx} f(x)g(x) = \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x) \). If we let \( u = f(x) \) and \( v = g(x) \),
then the product rule can be written in differential notation as \( d(\uv) = du \cdot v + dv \cdot u \).
Integrating both sides of this equation produces \( \uv = \int v du + \int u dv \) and solving for the
second integral produces the Integration by Parts Formula:
\[
\uv = \int u dv = \uv - \int v du
\]
We will do more work with this important method of integration after we finish
developing integration rules for all the basic elementary functions.

**EXPONENTIAL AND LOG FUNCTION INTEGRALS**

\[
\int e^x \, dx = e^x + c \quad \text{because} \quad \frac{d}{dx} e^x = e^x
\]

\[
\int a^x \, dx = a^x \ln a + c \quad \text{because} \quad \frac{d}{dx} a^x = a^x \ln a
\]

Now we know \( \frac{d}{dx} \ln x = \frac{1}{x} \) for \( x > 0 \). Thus \( \int \frac{1}{x} \, dx = \ln x + c \). But what if \( x < 0 \)?

Then \(-x > 0\) and \( \int \frac{1}{-x} \, d(-x) = \ln(-x) + c \). Combining these results produces:

\[
\int \frac{1}{x} \, dx = \ln |x| + c
\]

The process of evaluating \( \int \ln x \, dx \) using parts is shown below.

\[
\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + c
\]

\( u = \ln x \quad dv = dx \)

\( du = \frac{1}{x} \, dx \quad v = x \) (the \( c \) is omitted because it will cancel out)

Checking by differentiation shows that \( \frac{d}{dx} [x \ln x - x + c] = 1 \cdot \ln x + \frac{1}{x} \cdot x - 1 = \ln x \).

Now that we know how to integrate \( \ln x \), we can integrate \( \log_b x \).

\[
\int \log_b x \, dx = \int \frac{\ln x}{\ln b} \, dx = \frac{1}{\ln b} [x \ln x - x] + c = x \log_b x - x \log_b e + c = x \log_b \frac{x}{e} + c
\]

**TRIG INTEGRALS**
\[ \int \sin x \, dx = -\cos x + c \quad \text{because} \quad \frac{d}{dx} \cos x = -\sin x \quad \frac{d}{dx} \sin x = \cos x \]

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du = -\ln|u| + c = -\ln\cos x + c \]

\[ u = \cos x \]
\[ du = -\sin x \, dx \]

**Exercise:** Show that \( \int \cot x \, dx = \ln|\sin x| + c \)

\[ \int \sec x \, dx = \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du = \ln|u| + c = \ln|\sec x + \tan x| + c \]

\[ u = \sec x + \tan x \]
\[ du = (\sec x \cdot \tan x + \sec^2 x) \, dx \]

**Exercise:** Show that \( \int \csc x \, dx = -\ln|\csc x + \cot x| + c. \)

**INVERSE TRIG INTEGRALS**

Integration of inverse trig functions is handled by integration by parts.

\[ \int \sin^{-1} x \, dx = x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} \, dx = x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} \, du \]

\[ u = \sin^{-1} x \quad dv = dx \]
\[ u = 1 - x^2 \quad \Rightarrow \quad x \sin^{-1} x + \frac{1}{2} \frac{u^{1/2}}{1/2} + c \]

\[ du = \frac{1}{\sqrt{1-x^2}} \, dx \quad v = x \quad du = -2x \, dx \quad \Rightarrow \quad x \sin^{-1} x + \sqrt{1-x^2} + c \]

Integration of other inverse trig functions is handled in a similar manner.

**MORE INTEGRATION BY PARTS**
Chapter 9: Introduction to Calculus

As mentioned previously, parts is the second of the two major methods of integration (substitution being the other). This technique is frequently employed when integrating products of functions.

**Example 1:**
\[ \int x \sin x \, dx = -x \cos x + \int \cos x \, dx \]

\[ u = x \quad dv = \sin x \, dx \quad = -x \cos x + \sin x + c \]

\[ du = dx \quad v = -\cos x \]

**Example 2:**
\[ \int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \]

\[ u = x^2 \quad dv = \sin x \, dx \quad u = 2x \quad dv = \cos x \, dx \quad = -x^2 \cos x + 2x \sin x + 2 \cos x + c \]

\[ du = 2x \, dx \quad v = -\cos x \quad du = 2 \, dx \quad v = \sin x \]

Observe that in the examples above, we let \( x \) and then \( x^2 = u \). This choice was made because the degree of \( x \) in the \( du \) expression decreases, since \( u \) is polynomial. But what if neither of the functions in the product is polynomial? For example consider:

\[ \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \]

\[ u = e^x \quad dv = \sin x \, dx \quad u = e^x \quad dv = \cos x \, dx \]

\[ du = e^x \, dx \quad v = -\cos x \quad du = e^x \, dx \quad v = \sin x \]

Thus \( 2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x \) and \( \int e^x \sin x \, dx = \frac{1}{2} e^x \left( \sin x - \cos x \right) + c \)

**Note:** We integrated without really integrating! TotTB!

**TRIG POWERS INTEGRALS**

Next we will evaluate a few integrals involving powers of sine and cosine.

First recall that \( \cos 2x = 1 - 2 \sin^2 x \Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \).

Then \[ \int \sin^2 x \, dx = \int \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + c \]

**Exercise:** Show that \( \int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + c \).
Now consider an odd power.

\[ \int \sin^3 x \, dx = \int (\sin^2 x)(\sin x) \, dx = \int (-\cos^2 x)\sin x \, dx \]

\[ u = \cos x, \, du = -\sin x \, dx \Rightarrow \]
the integral = \[- \int (-u^2) \, du = -u + \frac{1}{3}u^3 + c = -\cos x + \frac{1}{3}\cos^3 x + c \]

**Exercise:** Show that \[ \int \cos^3 x \, dx = \sin x - \frac{1}{3}\sin^3 x + c \]

Prior to the widespread use of calculators and computer algebra (and calculus) systems, considerable time was spent developing formulas for integrating various powers of and/or combinations of trig functions. Many of the generalized results for integrals of trig functions as well as other elementary function integrals were listed in tables of integrals which sometimes stretched to over 500 formulas (which in one book was (tongue in cheek?) entitled a brief table of integral formulas). The modern approach today is to not spend as much time with integration techniques, so we will not do any more work with powers of trig functions.

**TRIG SUBSTITUTIONS**

In the last section we developed derivative formulas for inverse trig functions. These formulas' integral equivalents actually have more utility. For example, since \[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \]
we know \[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c. \] Likewise you can "see" other
inverse trig integral formula, and thus could extend your list of integration formula. But we won't do this.

All integrals similar to those which would be in this list extension can be evaluated by trig substitutions which are based on the fundamental Pythagorean trig identity \( \sin^2 x + \cos^2 x = 1 \). This identity can be rewritten as (1) \( 1 - \sin^2 x = \cos^2 x \) or (2) \( 1 + \tan^2 x = \sec^2 x \) or (3) \( \sec^2 x - 1 = \tan^2 x \). The chart below shows how to handle an integral which involves one of the quadratic binomial expressions \( a^2 - u^2 \), \( a^2 + u^2 \) or \( u^2 - a^2 \), where \( u \) is a differentiable function of \( x \). The object is to convert a binomial to a monomial.

For \( a^2 - u^2 \), let \( u = a \sin w \). Then \( a^2 - u^2 = a^2 - a^2 \sin^2 w = a^2(1 - \sin^2 w) = a^2 \cos^2 w \).

For \( a^2 + u^2 \), let \( u = a \tan w \). Then \( a^2 + u^2 = a^2 + a^2 \tan^2 w = a^2(1 + \tan^2 w) = a^2 \sec^2 w \).

For \( u^2 - a^2 \), let \( u = a \sec w \). Then \( u^2 - a^2 = a^2 \sec^2 w - a^2 = a^2(\sec^2 w) - 1) = a^2 \tan^2 w \).

For example, consider \( \int \frac{1}{9 + 4x^2} \, dx \). Let \( 2x = 3 \tan w \). Then \( 2 \, dx = 3 \sec^2 w \) and we obtain \( \int \frac{1}{9 + 9\tan^2 w} \frac{3}{2} \sec^2 w \, dw = \frac{1}{6} \int \frac{1}{w} \, dw + c = \frac{1}{6} \tan^{-1} \frac{2x}{3} + c \).

Note: The cofunction substitutions could be made, but these substitutions are commonly used.

The preceding substitution technique can also be used with quadratic trinomial expressions as shown in the example below.

\[
\int \frac{1}{3x^2 - 6x + 5} \, dx = \int \frac{1}{3(x-1)^2 + 2} \, dx = \int \frac{1}{2\tan^2 w + 2} \sqrt{3} \, \sec^2 w \, dw = \int \frac{1}{\sqrt{6}} \, dw
\]

Let \( \sqrt{3}(x-1) = \sqrt{2} \tan w \), \( \sqrt{3} \, dx = \sqrt{2} \sec^2 w \, dw \), \( \frac{1}{\sqrt{6}} \, w + c = \frac{1}{\sqrt{6}} \tan^{-1} \frac{3}{2}(x-1) + c \).

Note: Notice that the integral in the proceeding example is a rational function. Factorable rational expressions can be integrated by the method of Partial Fractions, but today many books, including this one, choose to omit this interesting, but tedious technique.

There are, as the note above suggests, other techniques of integration, but the two major techniques of substitution and parts are all we will examine in this book. Many integrals can not be evaluated exactly by any known technique. Such integrals can be approximated by numerical methods, some of which involve infinite series, the topic of the last section of this chapter. In the next section we will examine a few applications of integration. We conclude this section by applying the concept of using a definite integral to calculate area, in discussing what are called improper integrals.
IMPROPER INTEGRALS

An integral of the form \( \int_a^b f(x) \, dx \), is called improper if either its limit domain, [a,b], is not finite or the range of the integrand f(x) is not finite over its limit domain. To see why integrals of this type need to be handled carefully, consider the problem of finding the area "under the graph" of \( y = \frac{1}{x^2} \) from \( x = -1 \) to \( x = 2 \) as pictured below.

\[
\int_{-1}^{2} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_{-1}^{2} = -\frac{3}{2} \quad \text{which is clearly not the area.}
\]

In the process of attempting to find the area above, we must not integrate through a limit value where the integrand is undefined. Instead, we represent the value where the integrand is undefined by some letter such as \( b \) and compute:

\[
\lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x^2} \, dx + \lim_{b \to 0^+} \int_{b}^{2} \frac{1}{x^2} \, dx = \infty \quad \text{which is clearly a more satisfying result.}
\]

The other type of an improper integral is handled in the same manner. For example, suppose we want to find the area from \( 2 \) to \( \infty \) under the curve seen above. Then:

\[
\int_{2}^{\infty} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[-\frac{1}{x}\right]_{2}^{b} = \lim_{b \to \infty} \left( -\frac{1}{b} + \frac{1}{2} \right) = \frac{1}{2}.
\]

Problem Set 9.5

1. Express \( \lim_{\|P\| \to 0} \sum_{k=1}^{n} \left[ c_k^2 - 3c_k \right] \Delta x_k \), where \( P \) is a partition of [1,2] as a definite integral and evaluate.

Evaluate the following integrals [ 2 - 6]:
2. $\int (2x^2 - 5)^2 \, dx$  
3. $\int (2x - 5)^{10} \, dx$  
4. $\int_{0}^{\pi/4} \tan 2x \, dx$  
5. $\int \cos 5x \cos 3x \, dx$  
6. $\int_{0}^{\pi/3} \frac{1}{\sqrt{1 - \cos x}} \, dx$  

7. Find $\frac{dy}{dx}$ if $y = \int_{\sec x}^{2} \frac{1}{t^2 + 1} \, dt$  

Evaluate the following integrals [8 - 17]:  

8. $\int_{2}^{16} \frac{1}{2x\sqrt{\ln x}} \, dx$  
9. $\int_{\ln 4}^{\ln 9} e^{x/2} \, dx$  
10. $\int_{1}^{e} \frac{3\ln t}{t} \, dt$  
11. $\int_{-1}^{0} \frac{1}{\sqrt{3 - 2x - x^2}} \, dx$  
12. $\int_{0}^{1} \frac{x^3}{\sqrt{4 + x^2}} \, dx$  

[Hint: Complete the square on x.]  

13. $\int \frac{\cos x + \sin x}{\sin 2x} \, dx$  
14. $\int \frac{1}{\sqrt{(5 - 4x - x^2)^5}} \, dx$  
15. $\int \frac{1}{x^4\sqrt{x^2 - 2}} \, dx$  
16. $\int \frac{2x + 1}{4x^2 + 12x - 7} \, dx$  
17. $\int x^2 \ln x \, dx$
9.6 APPLICATIONS OF THE INTEGRAL

There are many applications of integration, but we will examine only five of them. First we will examine the process of solving differential equations which employs indefinite integration. Then we will look at four geometric applications of definite integration; those involving computing area, length of curve, volume and surface area.

DIFFERENTIAL EQUATIONS

Differential equations have been discussed several times previously. In this subsection, we will employ integration to solve a few types of DE's. First we consider the DE that produced the exponential growth/decay formula used in Section 7.4.

\[
\frac{dA}{dt} = kA \quad \Rightarrow \quad \frac{dA}{A} = k \, dt \quad \Rightarrow \quad \ln A = \int k \, dt + C
\]

Changing to exponential form produces

\[
A = e^{k t + C} = e^C e^{k t}
\]

When \( t = 0 \), \( A_0 = e^C e^{k \cdot 0} = e^C \)

And thus \( A = A_0 e^{k t} \) as seen previously.

Next consider the Ballistics problem seen in Section 9.4. An object is projected upward from the surface of the Earth, at an initial velocity of 160 ft/sec. It reaches an altitude of \( s = 160t - 16t^2 \) ft in \( t \) sec.

(a) How high does it reach? (b) With what velocity does it strike the ground?

Where does this position equation come from? Let an equation of the position of some object in motion as a function of time be given by \( s = f(t) \). Recall that the instantaneous rate of change of position as a function of time, which is called velocity, is given by \( v = \frac{ds}{dt} = f'(t) \). In turn, the instantaneous rate of change of velocity as a function of time, which is called acceleration, is given by \( a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t) \). Now for falling objects near the surface of the Earth, the acceleration due to the force of gravity exerted down towards the Earth is 32 ft/sec\(^2\). Thus \( a = -32 \).

\[
a = -32 \quad \Rightarrow \quad \frac{dv}{dt} = -32 \quad \Rightarrow \quad dv = -32 \, dt \quad \Rightarrow \quad \int dv = \int -32 \, dt \quad \Rightarrow \quad v = -32t + C
\]

Now when \( t = 0 \), then \( v = 160 \) ft/sec. Hence \( v = -32 \, t + 160 \)

\[
\begin{align*}
v &= -32t + 160 \quad \Rightarrow \quad \frac{ds}{dt} &= -32t + 160 \quad \Rightarrow \quad ds = (-32t + 160) \, dt \quad \Rightarrow \quad \int ds &= \int (-32 \, t + 160) \, dt \quad \Rightarrow \\
s &= -16t^2 + 160t + C.
\end{align*}
\]

Now when \( t = 0, s = 0 \). Hence \( s = -16t^2 + 160t \).
Thus you see that equations for velocity and, subsequently, distance (which is sometimes called displacement) are developed starting with integration of some given acceleration. This notion suggests three other notions. First, the constant of acceleration due to gravity depends on the planet, moon or other celestial object. Second, acceleration could result from other phenomena, such as a car accelerating down a road or decelerating (negative accelerating) to a stop or slower speed (positive velocity if going forward and negative velocity if going in reverse). Third, acceleration need not be constant, in which case \( \frac{da}{dt} \neq 0 \). In ballistics \( \frac{da}{dt} \) is appropriately called \textbf{jerk}.

Both of the developments seen above involved solving differential equations. The DE’s we worked with all belong to a class of DE’s called \textbf{Separable}. In this type of DE, it is possible to combine two variables with their differentials on separate sides of an equation and then integrate each side of the equation. When this can not be done, other methods have to be employed. A course in DE consists of a carefully sequenced series of DE types and corresponding solution techniques. We will conclude this subsection with an examination of the solution technique for one type of DE besides the separable type we have just discussed.

The equation \( \frac{dy}{dx} + Py = Q \) is a first order, first degree DE.

Let an \textbf{integrating factor} \( r = P(x) \) such that \( r \frac{dy}{dx} + rPy = \frac{dy}{dx}y + \frac{dy}{dx}r \)

Then \( \frac{dr}{dx} = rP \) and so \( \frac{dr}{dx} = rP \)

Solving this separable DE yields:

\[
   \frac{dr}{r} = P \, dx \quad \Rightarrow \quad \ln |r| = \int P \, dx + C \quad \Rightarrow \quad |r| = e^{\int P \, dx} = e^C \cdot e^{\int P \, dx}
\]

Thus \( r = \pm ce^{\int P \, dx} \) since we do not need the most general integrating factor.

So making this substitution produces \( \frac{dr}{dx} (ry) = rQ \) for which the solution is:

\[
   ry = \int rQ \, dx + C
\]

**Example:** Consider \( x \frac{dy}{dx} - 3y = x^2 \). In standard form this is \( \frac{dy}{dx} - \frac{2}{x} y = x \).

So \( P = -\frac{2}{x} \) and \( Q = x \). \( r = e^{-\int P \, dx} = e^{-3 \ln x} = \frac{1}{x^3} \)

Thus \( \frac{1}{x^3} y = \int \frac{1}{x^3} \, dx + C = -\frac{1}{x} + C \) and \( y = -x^2 + C \cdot x^3 \)

**AREA**

We have seen how the area under the graph of a positive valued function \( f(x) \) from \( a \) to \( b \) can be computed by Riemann sums and by the integral \( \int_a^b f(x) \, dx \) which equals \( F(b) - F(a) \). We have also generated formulas for a sizable collection of anti derivatives (indefinite integrals). Shortly we will developing the process of calculating lengths of
curves, volumes and surface areas for various solids. The integration formulas for all of
these geometric applications of integration can be developed in essentially the same
manner as we previously saw for positive valued functions. In this book we will set up,
and then solve, DE's to generate the integration formulas.

Consider a function \( f \) continuous on \([a,b]\), and let \( x \in (a,b) \). Denote the area under \( f \)
from \( a \) to \( x \) by \( A^x_a \). Clearly as \( x \) changes, \( A^x_a \) changes, so \( A^x_a \) is some function of \( x \), call
it \( g(x) \). Observe that \( g(a) = 0 \) since \( A^a_a = 0 \). Also in the drawing \( f(x) > 0 \) on \((a,b)\).
Hence our area function \( g(x) \), is an increasing function on \((a,b)\). Suppose \( x \) changes by
some increment \( \Delta x \). Then \( A^x_a \) will change by some increment we will call \( \Delta A^x_a \).

\[
\Delta A^x_a = A^{x+\Delta x}_a - A^x_a = f(c) \Delta x \text{ where } x < c < x + \Delta x.
\]

This may be interpreted as the area of a rectangle with width \( \Delta x \) and height \( f(c) \). We
know there is such a rectangle because \( f(m) \Delta x \leq A^{x+\Delta x}_x \leq f(M) \), where \( f(m) \) is the min
and \( f(M) \) the max functional value on \([x,x+\Delta x]\), and \( f \) takes on all values between these
two values (intermediate value theorem).

\[
\Delta A^x_a = f(c) \Delta x \quad \Rightarrow \quad \frac{\Delta A^x_a}{\Delta x} = f(c)
\]

\[
\lim_{\Delta x \to 0} \frac{\Delta A^x_a}{\Delta x} = \lim_{\Delta x \to 0} f(c) \quad \Rightarrow \quad \frac{dA^x_a}{dx} = f(x)
\]

\[
\int dA^x_a = \int f(x) \, dx \quad \Rightarrow \quad A^x_a = F(x) + C \quad \text{where } F'(x) = f(x)
\]

Now if \( x = a \), then since \( A^a_a = 0 \), we have \( 0 = F(a) + C \quad \Rightarrow \quad C = - F(a) \)

Hence \( A^b_a = F(x) - F(a) \) and thus \( A^b_a = F(b) - F(a) \), which equals \( \int_a^b f(x) \, dx \) as we have
previously seen.

Now consider finding the area between two curves depicted below.
The area under \( f(x) \) is \( \int_a^b f(x) \, dx \) and the area under \( g(x) \) is \( \int_a^b g(x) \, dx \).

The area between the curves is \( \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx \).

Thus to find the area between curves, simply subtract the lower curve function from the upper curve function and integrate. This technique works regardless of whether both curves are above the x-axis; one is above and the other is below the x-axis; or both are below the x-axis. (You should convince yourself that this is true.)

Now consider the picture below.

Representative rectangles are drawn to show how to calculate the height expression. The area bounded between the two curves \( A = \int_a^b [f(x) - g(x)] \, dx + \int_b^c [g(x) - f(x)] \, dx \).

Observe that all you need to remember is to subtract the lower curve equation from the upper curve equation and then integrate.

As a final note concerning area, we observe that for a positive valued function \( f(x) \), \( \int_a^b f(x) \, dx = \int_a^b [f(x) - 0] \, dx \), which is the area bounded by \( y = f(x) \) and \( y = 0 \) (the x-axis) over the interval \([a, b]\). Thus our original area formula \( \int_a^b f(x) \, dx \) is a special case of the more general notation of area between curves. You might also realize that for a region below the x-axis, the upper curve would be \( y = 0 \), and the area would be...
calculated by \( \int_{a}^{b} [0-f(x)] \, dx \), which equals \(-\int_{a}^{b} f(x) \, dx\). Thus it becomes easy to understand why in this case, \( \int_{a}^{b} f(x) \, dx \) would be negative.

**Exercise:** Explain why for a positive valued function \( f(x) \) on \([a,b]\),
\[
\int_{b}^{a} f(x) \, dx < 0.
\]

As an application of our area formula, we will show that the area enclosed by a circle with radius \( r \) is given by \( \pi r^2 \). Shown below is the graph of \( x^2 + y^2 = r^2 \). A representative rectangle is drawn in one quarter of the circle

Solving the equation for \( y \) produces \( y = \pm \sqrt{r^2-x^2} \). The height of the rectangle is given by \( \sqrt{r^2-x^2} \). So the area enclosed by the circle is \( 4 \int_{0}^{r} \sqrt{r^2-x^2} \, dx \). The evaluation of this integral is shown below.

\[
4 \int_{0}^{r} \sqrt{r^2-x^2} \, dx = 4 \int_{0}^{\pi/2} \sqrt{r^2-r^2 \sin^2 \theta} \, r \cos \theta \, d\theta \\
\theta = x = r \sin \theta \\
dx = r \cos \theta \, d\theta \\
dx = 4 \int_{0}^{\pi/2} \sqrt{r^2 \cos^2 \theta} \, d\theta \\
= 4 \int_{0}^{\pi/2} r^2 \cos^2 \theta \, d\theta \\
= 4 \int_{0}^{\pi/2} \frac{r^2}{2} (1+\cos 2\theta) \, d\theta \\
= 2r^2 \left( \frac{\theta + \frac{1}{2} \sin 2\theta}{} \right)_{0}^{\pi/2} \\
= 2r^2 \left( \left[ \frac{\pi}{4} + 0 \right] - \left[ 0 - 0 \right] \right) \\
= \pi r^2
\]

**Note:** You noticed that, as part of the substitution process, we changed the limits when we changed the variable. Although we could go back to the original variable before substituting in the limits, I think it is just as easy, and also safer to do the work in this manner.
In the development and the example shown in this subsection, we worked with functions of the form \( y = f(x) \). The area between curves with equations of the form \( x = g(y) \) can be found in a similar manner.

**Exercise:** Solve for \( x \) in terms of \( y \) in the circle equation seen above and then show the area of the circle is still \( \pi r^2 \). Hint: Your limits interval and differential will all be in terms of \( y \).

---

### AVERAGE VALUE

Related to the area concept is the concept of **Average Value**. The average value of a function over some interval \([a,b]\) is defined as \( \text{Ave. Value} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \). In other words, average value is simply the height (which could be negative) of a rectangle on an interval \([a,b]\) which has the same area as the region(s) bounded between a curve and the \( x \)-axis.

**Example:** The average value of \( y = \sin x \) on \([0, \frac{3\pi}{2}]\) is

\[
\frac{1}{3\pi/2} \int_{0}^{3\pi/2} \sin x \, dx = \frac{2}{3\pi} \left[ -\cos x \right]_{0}^{3\pi/2} = \frac{2}{3\pi}
\]

The US. Weather Bureau uses the concept of average value to calculate the average temperature for a period of time. A piece of graph paper is attached around a cylinder which makes one complete revolution in some period of time such as a day. A stylus attached to a thermometer traces out a graph of the temperature on the graph paper. The area under the graph divided by the time period is the average temperature for that time period. Average value is a simple, yet very useful concept.
AREAS IN POLAR COORDINATES

First, recall that the area of a sector of a circle is given by \( A_{\text{sector}} = \frac{1}{2} r^2 \theta \) where \( r \) is the radius of the circle and \( \theta \) is the angle of the sector. Now consider the picture below.

We want to find a formula for the area of the sector of the region bounded by the polar graph \( r = f(\theta) \) from \( \theta = a \) to \( \theta = b \). We will use the same procedure that we used previously.

Let \( \Delta \theta \) be the angle of the sector of the region bounded by the polar graph from \( \theta = a \) to \( \theta = b \), from \( \theta \) to \( \theta + \Delta \theta \). We know there is some \( \theta^* \) in the interval from \( \theta \) to \( \theta + \Delta \theta \) such that the area of a sector with angle \( \Delta \theta \) taken from a circle with radius \( f(\theta^*) \) is equal to the area of the sector from \( \theta \) to \( \theta + \Delta \theta \) bounded by the polar graph. Thus:

\[
A_{\theta + \Delta \theta} = \Delta A_{\theta} = \frac{1}{2} f^2(\theta^*) \Delta \theta \quad \Rightarrow \quad \frac{\Delta A}{\Delta \theta} = \frac{1}{2} f^2(\theta^*) \Rightarrow \frac{dA}{d\theta} = \frac{1}{2} f^2(\theta)
\]

Solving this DE produces the desired formula, \( A_a^b = \int_a^b \frac{1}{2} f^2(\theta) \, d\theta \).

**Exercise:** Show that the area inside the cardioid \( r = 1 + \sin \theta \) is \( \frac{3\pi}{2} \).
LENGTH OF CURVE

In this subsection, we will develop the formula for the length of a curve.

\[
\Delta L^y_a = L^y_{x+\Delta x} \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \Delta x \Rightarrow \frac{\Delta L^y_a}{\Delta x} \approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}
\]

As \( \Delta x \to 0, \Delta y \to 0 \) and \( \Delta L \to 0 \) and we have

\[
dL^y_a = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \Rightarrow \int dL^x_a = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \Rightarrow L^x_a = G(x) + C \quad \text{where } G(x) = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

Now \( L^a_a = 0 \Rightarrow G(a) + C = 0 \Rightarrow C = -G(a) \). Thus, \( L^x_a = G(x) - G(a) \) and so

\[
L^b_a = G(b) - G(a) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

In the formula shown above, \( a \) and \( b \) are the limits, and \( x \) is the variable of integration. Suppose \( x \) is a function of \( y \). In the development above, we factored out \( \Delta x \) and then took the limit as \( \Delta x \to 0 \). If \( x = f(y) \), we would factor out \( \Delta y \) and then go through the same steps as before (using \( c \) and \( d \) as the limits) producing:

\[
L^d_c = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy.
\]
Exercise: If the curve is defined parametrically by \( x = f(t) \) and \( y = g(t) \), then show that the Length of Curve formula is

\[
L_A^B = \int_A^B \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

Note: The parameter values of \( t \) that generate points A and B are used for the limits.

All three of these length formulas can be expressed as a single general formula as:

\[
L_A^B = \int_A^B dL
\]

where \( dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \), \( \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy \), or \( \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \) as appropriate.

Note: \( dL \) is often referred to as the differential of arc length.

As an example of the process of finding length, we will show that the circumference of our circle with radius \( r \) is given by \( 2\pi r \).

For the upper semicircle, \( y = \sqrt{r^2 - x^2} \). \( \frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}} \)

\[
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}}
\]

\[
C = 4r \int_0^r \frac{1}{\sqrt{r^2 - x^2}} \, dx = 4r \int_0^{\pi/2} \frac{d\theta}{\sqrt{r^2 - r^2 \sin^2 \theta}} = 4r \int_0^{\pi/2} \frac{d\theta}{r} = \frac{2\pi r}{2} = \pi r
\]

LENGTH OF A POLAR CURVE
To generate the formula for length when a polar equation \( r = f(\theta), \ a \leq \theta \leq b \), is given, just parametrize the equation by letting:

\[
x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta.
\]

Then \( \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta \) and \( \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta \).

Hence \( \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2 \theta - 2r \frac{dr}{d\theta} \cos \theta \sin \theta + r^2 \sin^2 \theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \).

Now, combining terms and using the fundamental Pythagorean Trig identity,

\[
\sin^2 \theta + \cos^2 \theta = 1,
\]

yields \( \left(\frac{dr}{d\theta}\right)^2 + r^2 \).

Hence the parametric length formula \( L_A^B = \int_A^B \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt \), becomes:

\[
L_a^b = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ d\theta.
\]

**Exercise:** Show that the length of the cardioid \( r = 1 + \sin \theta \) is 8.

**Hint:** You will need to use a trig substitution.
In this book we will examine only one type of volume - that which is generated by revolving some planer region about a line. There are three methods for finding volumes of solids of revolution - discs, washers and shells.

**The Disc Method**

Shown below is a region bounded by \( y = f(x) \), \( y = 0 \), \( x = a \) and \( x = b \).

Imagine this region being revolved about the x-axis and then sliced into vertical disk-like pieces. Each piece has volume equal to the volume generated by revolving a strip under the curve from \( P \) to \( Q \) about the x-axis. Now we have already seen that each strip area is equal to the area of a rectangle with the same width as the strip and a height equal to the functional value of some number in the strip width interval. Thus, the volume of a rotated strip equals the volume of a rotated rectangle. Now when a rectangle is rotated, it sweeps out a cylinder (which is called a disc when the radius of the cylinder is large in comparison to the height of the cylinder).

Similar to what we have done previously (with some steps omitted),

\[
\Delta V_x^a = V_x^{a+\Delta x} = \pi f^2(c) \Delta x \quad \text{where} \quad x < c < x + \Delta x
\]

\[
\frac{dV_x^a}{dx} = \pi f^2(x) \quad \Rightarrow \quad \int dV_x^a = \int \pi f^2(x) \, dx \quad \Rightarrow \quad V_x^a = G(x) + C \quad \text{where} \quad G(x) = \int \pi f^2(x) \, dx
\]

Now \( V_x^a = 0 \) \( \Rightarrow \) \( G(a) + C = 0 \) \( \Rightarrow \) \( C = -G(a) \). Thus, \( V_x^a = G(x) - G(a) \) and so

\[
V_x^b = G(b) - G(a) = \int_a^b \pi f^2(x) \, dx
\]
Now we illustrate finding this type of volume by showing that the volume of a sphere with radius $r$ is given by $\frac{4}{3} \pi r^3$.

\[
V = 2 \int_0^r \pi \left( r^2 - x^2 \right)^2 dx = 2\pi \int_0^r \left( r^2 - x^2 \right) dx = 2\pi \left[ r^2 x - \frac{1}{3} x^3 \right]_0^r = 2\pi \left[ r^3 - \frac{1}{3} r^3 \right] = \frac{4}{3} \pi r^3
\]

Exercise: Show that the formula for finding the volume generated by revolving the region bounded by $x = g(y)$, $x = 0$, $y = c$, and $y = d$ about the $y$-axis is given by $\int_c^d \pi g^2(y) dy$ and then show, using this formula, that the volume of a sphere with radius $r$ is $\frac{4}{3} \pi r^3$.

**The Washer Method**

Now suppose we generate a solid by revolving a region between curves about the $x$-axis as shown below.
Analogous to subtracting areas when finding area between curves, the volume produced can be found by subtracting volumes.

The volume produced by revolving \( f(x) \) is \( \int_a^b \pi f^2(x) \, dx \) and volume produced by revolving \( g(x) \) is \( \int_a^b \pi g^2(x) \, dx \).

Thus the volume of the solid is \( \int_a^b \pi f^2(x) \, dx - \int_a^b \pi g^2(x) \, dx = \pi \int_a^b [f^2(x) - g^2(x)] \, dx \).

Volumes generated by revolving about a vertical line are calculated similarly.

**Question:** Why do you think this technique is called the washer method?

---

**The Shell Method**

Consider revolving the region bounded by \( y = f(x) \) and the x-axis shown below about the y-axis.

Perhaps we could find \( x \) as two functions of \( y \) and use the washer method, but we might not be able to or want to do this. Let \( V_a^x \) = the volume generated by revolving the region from \( a \) to \( x \) about the y-axis. Let \( V_x^{x+\Delta x} \) equal the volume generated by revolving the strip from \( x \) to \( x + \Delta x \) about the y-axis. We have seen before that there is a \( c \) in \( (x, x+\Delta x) \) such that the area of the strip equals the area of a rectangle \((f(c)\Delta x)\) with the same width \( \Delta x \). Hence the volume swept out by revolving the strip about the y-axis would be equal to the volume swept out by revolving the rectangle about the y-axis. This solid is a circular wall. The volume of this wall is equal to the base area time the height. The base area is the area between two concentric circles as shown below.
The base area = $\pi(x + \Delta x)^2 - \pi x^2 = \pi(2x + \Delta x)\Delta x$ and the height of the wall is $f(c)$.

Thus $\Delta V_a^x = \pi(2x + \Delta x)\Delta x \cdot f(c)$ and $\frac{\Delta V_a^x}{\Delta x} = \pi(2x + \Delta x) f(c)$

Which, as $\Delta x \to 0$, becomes the DE $\frac{dV_a^x}{dx} = 2\pi x f(x)$.

Thus, similar to our previous developments:

$$V_a^h = \int_a^b 2\pi x f(x) \, dx$$

This formula is easily remembered as $V_A^B = \int_A^B 2\pi r h t$ using the picture below.

$r$ = distance of a point on the curve from the axis of rotation.

$h$= height of a representative rectangle to be rotated

$t$ = thickness of a representative rectangle to be rotated (dx for a horizontal rotation)

Perhaps you noticed a significant difference between the disc and shell method. When using the disc method on a solid produced by a rotation about a horizontal axis, the variable of integration is $x$, whereas when using the disc method on a solid produced by a rotation about a vertical axis, the variable of integration is $y$. This is reversed with the shell method. When using the shell method on a solid produced by a rotation about a horizontal axis, the variable of integration is $y$, whereas when using the shell method on a solid produced by a rotation about a vertical axis, the variable of integration is $x$. 

The table below shows the preferred method for various problem types

<table>
<thead>
<tr>
<th>Equation</th>
<th>Axis</th>
<th>Preferred Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = f(x)</td>
<td>horizontal</td>
<td>Disc</td>
</tr>
<tr>
<td>y = f(x)</td>
<td>vertical</td>
<td>Shell</td>
</tr>
<tr>
<td>x = g(y)</td>
<td>horizontal</td>
<td>Shell</td>
</tr>
<tr>
<td>x = g(y)</td>
<td>vertical</td>
<td>Disc</td>
</tr>
</tbody>
</table>

**Exercise 1:** Find the volume generated when the right semicircle of the graph of \( x^2 + y^2 = r^2 \) is rotated about the y-axis.

**Exercise 2:** A shape called a torus is generated by revolving a circular disc about a line. Find the volume of the torus generated by revolving the circular disc defined by \( x^2 + y^2 \leq r^2 \) about the line \( x = R \) where \( R > r \).
SURFACE AREA

Before developing the integral formula for finding the surface area of a solid of revolution, we need to review a few results from geometry. If a cone such as the one shown on the left below, were cut along PT and laid out flat, a sector, as pictured on the right below, would be formed.

Observe that the radius of the sector is equal to the slant height $l$ of the cone and the arc of the sector is equal to the circumference of the cone, $2\pi r$. Now the ratio of the arc of the sector to the circumference of the circle, of which the sector is a part, equals the ratio of the angle of the sector to the angle of the circle. Likewise, the ratio of the area of the sector to the area of the circle equals the ratio of the angle of the sector to the angle of the circle.

Thus: \[ \frac{2\pi r}{2\pi l} = \frac{\theta}{2\pi} \] and \[ \frac{A_{\text{sector}}}{\pi l^2} = \frac{\theta}{2\pi} \]

Hence: \[ A_{\text{sector}} = \frac{\theta}{2\pi} \cdot \pi l^2 = \frac{2\pi r}{2\pi l} \cdot \pi l^2 = \pi rl \]

Since the area of the sector is the lateral surface of the cone, we have \[ LSA_{\text{cone}} = \pi rl. \]

Now consider the drawing below.

If the top cone is removed, what remains is called a frustum of a cone.

\[ LSA_{\text{frustum}} = LSA_{\text{entire cone}} - LSA_{\text{top cone}} = \pi l_1^2 - \pi r_1 l_1 \]

substituting \[ l_1 = l_2 - l \] produces \[ LSA_{\text{frustum}} = \pi (r_2 l_2 - r_1 (l_2 - l)) = \pi ((r_2 - r_1)l_2 + r_1 l) \]
By similar triangles:  \( \frac{r_2}{l_2} = \frac{r_2 - r_1}{l} \Rightarrow l_2 = \frac{r_2 l}{r_2 - r_1} \)

Substituting in produces:  \( LSA_{frustum} = \pi (r_2 l + r_1 l) = \pi (r_2 + r_1) l \)

Now if we call \( r_m \) the radius of the midsection of the frustum, then \( r_m = \frac{r_2 + r_1}{2} \) and our formula for the lateral surface area of the frustum is given by \( 2\pi r_m l \).

Now we are ready to develop the integral formula for calculating the surface area of a solid of revolution.

Shown below is a region bounded by \( y = f(x) \), \( y = 0 \), \( x = a \) and \( x = b \).

Let \( S^x_a \) denote the surface from \( a \) to \( x \), and \( \Delta S^x_a \) denote the change in the surface from \( a \) to \( x \).

\[
\Delta S^x_a = S^{x + \Delta x}_x \approx LSA_{frustum} \text{ from } x \text{ to } x + \Delta x = 2\pi \left( f(x) + \frac{f(x + \Delta x)}{2} \right) \sqrt{(\Delta x)^2 + (\Delta y)^2}
\]

\[
= \pi (f(x) + f(x + \Delta x)) \sqrt{1 + \left( \frac{\Delta y}{\Delta x} \right)^2} \Delta x
\]

Thus \( \frac{\Delta S^x_a}{\Delta x} \approx \pi \left( f(x) + f(x + \Delta x) \right) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \Delta x \)

Similarly to the development seen in the other applications, \( \Delta x \to 0 \Rightarrow \)

\[
\frac{dS^x_a}{dx} = 2\pi f(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \text{ and } A^b_a = \int_a^b 2\pi f(x) \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]
Similar to the Length of Curve formula, this formula for Surface Area is best remembered as \( A_a^b = \int_a^b 2\pi r \, dL \) where \( r \) is the expression for the radius from the axis of rotation to the curve, and \( dL \) is the differential of arc length, which is expressed as:

\[
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{or} \quad \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy \quad \text{or} \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
\]

As an example of finding surface area, we will show that the surface area of a sphere of radius \( r \) is given by \( 4\pi r^2 \).

As seen previously in the Length of Curve subsection, \( \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{r}{\sqrt{r^2 - x^2}} \).

Thus \( SA = 2 \int_0^r 2\pi \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} \, dx = 4\pi r \int_0^r dx = 4\pi r x \int_0^r dy = 4\pi r^2 \).

**Exercise:** Suppose the region between \( y = \frac{1}{x} \) and \( y = 0 \), for \( x \geq 1 \), is revolved about the x-axis as shown below.

This shape is known as **Gabriel's horn**. Show that Gabriel's horn has the interesting feature of having infinite surface area, yet finite volume.

There are many other applications of integration, but the five we examined provide a glimpse into the power and utility of this branch of calculus.
Problem Set 9.6

Solve the following DE’s [1 - 2].

1. \( x e^{y/x} + y - x \frac{dy}{dx} = 0 \)  
2. \( x \frac{dy}{dx} + y = \sin x \)

3. A car traveling at a rate of 60 mph (88 ft./sec.) has its brakes applied producing a constant deceleration of 44 ft./sec.\(^2\) \((a = -44)\). What distance will it cover before coming to a stop?

4. Find the area bounded by the positive x and y axes and the curve \( y = 4 - x^2 \) by: (a) Riemann sums and (b) integration.

5. Consider the graphs of \( y = 2x + b \) and \( y^2 = 4x \).
   (a) What is the set of values of \( b \) for which the graphs intersect in two distinct points?
   (b) If \( b = -4 \), find the area of the region enclosed.

6. Write an equation of (a) a parabola and (b) a cosine wave passing through \((-b,0)\), \((0,h)\) and \((b,0)\). Using integration, determine which of these along with the x-axis between \(-b\) and \(b\) would enclose the most area.

7. Sketch the graph of the region enclosed by the graphs of \( y = x^2 - x \) and \( y = \sin \pi x \), set up the integral that would calculate its area and find its exact area

8. Graph and find the area between the curves \( y = \ln x \) and \( y = \ln 2x \) from \( x = 1 \) to \( x = 5 \).

9. Find the average value of \( f(x) = x^2 - 2x \) over \( 1 \leq x \leq 3 \).
10. Give the polar equation of the curve shown below and then calculate the exact area enclosed by the curve.

11. Find the parametric form integral that calculates the area enclosed by the graph of $x = 2 \cos t, y = 1 - 2 \sin t$ on $[0, 2\pi]$ and then find the exact area.

12. Find the length of the astroid $x = 3 \cos^3 t, y = 3 \sin^3 t$.

13. Graph the circle $x^2 + y^2 = 16$ and then find the exact volume of the solid generated when the circle is revolved about (a) the $y$-axis and (b) the line $y = 6$.

14. Suppose that a football has a major diameter of 30 cm and a minor diameter of 20 cm. Suppose that half of the cross section is a region bounded between a parabola and the $x$-axis as shown below.

(a) Show that the equation of the parabola is $y = 10 - \frac{2}{45}x^2$.

(b) Show an integral that produces the volume of this math football and then use your calculator to find the exact volume.

15. Sketch the region bounded by the graphs of $y = x^2$ and $x = y^2$. Show an integral that produces the volume when the region is revolved about the line $y = -1$ and then use your calculator to do the integration to find the exact volume.

16. Graph the region bounded by $y = \frac{1}{\sqrt{x+1}}$, $y = 0$, $x = 0$ and $x = 1$ and then find the volume of the solid generated by revolving the region about the $x$-axis.
9.7 INTRODUCTION TO INFINITE SERIES

Previously it was mentioned that there exist many integrals for which no direct evaluation process exists. It was also mentioned that the integration of such integrals can be expressed as a series and then integrated term by term. In this section we will develop the process of finding series expansions of functions and also see how to find intervals of convergence for series.

Many mathematicians, over many centuries, worked with series. Consider the function \( \frac{1}{1-x} \). By long division:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.
\]

From the example above, you can see that a series is a collection of terms added together. This series is called a power series because the terms all involve powers (of \( x \) in this case). Now, in the power series seen above, suppose we let \( x = 2 \). Then we have the obviously incorrect equation:

\[
\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + 16 + \cdots
\]

Next recall the Binomial Theorem seen previously.

\[
(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n
\]

The Generalized Binomial Theorem says that the Binomial Theorem holds regardless of the exponent type. So expressing \( \frac{1}{1-x} \) as \( (1-x)^{-1} \), expanding, and again substituting in 2 for \( x \) produces the same incorrect equation:

\[
(1-2)^{-1} = 1^{-1} + (-1)1^{-2}(-2)^1 + \frac{(-1)(-2)}{2}1^{-3}(-2)^2 + \frac{(-1)(-2)(-3)}{3!}1^{-4}(-2)^3 + \cdots
\]

\[
= 1 + 2 + 4 + 8 + 16 + \cdots
\]

**Exercise:** Show that \( \frac{x}{1-x} = x + x^2 + x^3 + \cdots \)

and that \( \frac{x}{x-1} = 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots \).

Now if these two equations are summed we have:

\[
\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots = \frac{x}{1-x} + \frac{x}{x-1} = 0
\]

which also can't be right.

These results both amazed and puzzled mathematicians and eventually they realized a need to systematically and rigorously examine and organize the plethora of ideas that had been developed during the Renaissance after the dark ages. This transitioning to modern mathematics included extensive work with infinite series and the related work with sequences.

SEQUENCE
Consider the function \( f(x) = x^2 \). If we restrict the domain to the set of natural numbers 1, 2, 3, \ldots, then the range values are 1, 4, 9, \ldots. This ordered set of values is an example of what is called a \textit{sequence}. Thus a sequence may be defined as a function whose domain is the set of natural numbers. Symbolically we write a sequence \( f \) as:
\[
\{ (n,f(n)) \mid n \in \mathbb{N} \text{ or in abbreviated form, just as } \{f(n)\}\}
\]
Now recall finding limits for functions as the variable approached plus or minus infinity. We can apply the same procedure to sequences and thus determine if a sequence does or does not have a limit. For example \( \left\{ \frac{1}{n} \right\} \rightarrow 0 \).

**SERIES**

If the terms of some sequence are added together, a series is formed. We can indicate a series using the sigma notation. For example:
\[
1 + 4 + 9 + 16 + 25 = \sum_{k=1}^{5} k^2.
\]
The series shown above is finite. On the other hand, a series such as:
\[
1 + 4 + 9 + 16 + 25 + \cdots = \sum_{k=1}^{\infty} k^2
\]
goes on forever and is called an \textit{infinite series}. Now it should be apparent that as \( n \rightarrow \infty \), the sum seen above grows without bound and thus we say this series diverges, which means it \textit{does not converge}. On the other hand, although growing, the series seen below does have a limiting value (which we will see later is 2), and is thus said to converge.

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}
\]

One way to deal with the problem of determining convergence or divergence of a series is to look at the sequence of partial sums \( \{S_n\} \) where \( S_n \) represents the sum of the first \( n \) terms of the series. If we let \( S \) represent the sum of all the terms of the series, then \( S = \lim_{n \to \infty} S_n \). Therefore, if \( \lim_{n \to \infty} S_n \) DNE, then \( S \) DNE, and we say the series is divergent.

Now with some thought, you would realize that the only way for the limit of the sequence of partial sums to exist is for the terms of the series to be approaching zero. That is to say, a series \( \sum_{n=1}^{\infty} a_n \) does not converge if \( \lim_{n \to \infty} a_n \neq 0 \).

For example, \( \sum_{n=1}^{\infty} \frac{n}{1+n} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \) does not converge because \( \lim_{n \to \infty} \frac{n}{1+n} = 1 \neq 0 \).

This test is appropriately called the \textit{n}th \textbf{term test for divergence}. Now, you should realize that a series could have \( \lim_{n \to \infty} a_n = 0 \) but still diverge. For example later we will
see that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent even though \( \lim_{n \to \infty} \left\{ \frac{1}{n} \right\} = 0 \). On the other hand, \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) has \( \lim_{n \to \infty} \left\{ \frac{1}{n^2} \right\} = 0 \) and is a convergent series.

Previously we discussed two special series - arithmetic and geometric. Recall that \( \sum_{n=1}^{\infty} [a + (n-1)d] \) is arithmetic and \( \sum_{n=1}^{\infty} ar^{n-1} \) is geometric. For the AS, \( S_n = \frac{n}{2} [2a + (n-1)d] \). By the n\textsuperscript{th} term test for divergence, the AS diverges (except for \( a = 0 = d \)). For the GS, \( S_n = \frac{a(1-r^n)}{1-r} \). Now if \(|r| < 1\), then \( \lim_{n \to \infty} s_n = \frac{a}{1-r} \), and thus \( S = \frac{a}{1-r} \) for \(|r| < 1\). If \(|r| \geq 1\), then the series diverges (except when \( a = 0 \)). This special class of series, for which convergence or divergence is well defined, is very useful as you will see later.

**THE INTEGRAL TEST**

Earlier in this section, it was mentioned that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent. Now we will see why this is true. Consider the graph of \( y = \frac{1}{x} \) for \( x > 0 \), shown below.

The area from 1 to \( \infty \) under the graph is given by the improper integral \( \int_{1}^{\infty} \frac{1}{x} \, dx \), which you should verify is undefined. Now the terms in the harmonic series can be interpreted as the areas of the circumscribed rectangles in the figure above. Thus, since the sum of the areas of the rectangles is greater than the area under the curve, we can conclude that the harmonic series is divergent.

Also, earlier in this section, it was mentioned that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent. Consider the graph of \( y = \frac{1}{x^2} \) for \( x > 0 \) shown below.
The area from 1 to $\infty$ under the graph is given by the improper integral $\int_1^{\infty} \frac{1}{x^2} \, dx$, which you should verify is equal to 1. Now the terms in the this series can be interpreted as the areas of the inscribed rectangles in the figure above. Thus, since the sum of the areas of the rectangles is less than $(1 + \text{the area from 1 to } \infty \text{ under the curve})$, we can conclude that this series is convergent.

Both of the preceding examples involved positive decreasing functions. If $f(x)$ is a positive decreasing function, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x) \, dx$ either both converge or both diverge. This result is known as the Integral Test.

**Exercise:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p series. Use the integral test to show that the series converges if $p > 1$ and diverges if $p \leq 1$.

**THE COMPARISON TEST**

At this point, we know when geometric or p series converge and diverge. Other series can be compared to these series. If a series $\sum_{n=1}^{\infty} a_n$ is convergent and another series $\sum_{n=1}^{\infty} b_n$ is term for term $\leq \sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ is also convergent. Likewise if $\sum_{n=1}^{\infty} a_n$ is divergent and term for term $\sum_{n=1}^{\infty} b_n \geq \sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ also diverges. This obvious result is know as the Comparison Test.
THE RATIO TEST

In some standard calculus texts considerable pages (over 20 for one text) are devoted to tests for convergence. We will present only one more test, called the ratio test, for a positive valued series. Consider the series \( \sum_{n=1}^\infty a_n \). The ratios \( \frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \ldots, \frac{a_{n+1}}{a_n} \) measure the growth ratio of a series. If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \), the positive valued series \( \sum_{n=1}^\infty a_n \) converges. If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1 \), the positive valued series \( \sum_{n=1}^\infty a_n \) diverges. If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \), then the test is inconclusive. A formal proof for this test can be found in a standard calculus text. Now we return to the task of finding series expansions for various functions.

POWER SERIES

A power series is a polynomial of the form \( k_0 + k_1 x + k_2 x^2 + \cdots \) that can be used to represent a function \( f(x) \). We will actually find our power series in the form \( k_0 + k_1 (x-a) + k_2 (x-a)^2 + \cdots \) which is said to be a polynomial expansion about \( a \).

Observe that \( \int_a^b f'(t) \, dt = f(b) - f(a) \), and hence \( f(b) = f(a) + \int_a^b f'(t) \, dt \).

We will use integration by parts repeatedly on the integral in the formula above to arrive at our power series.

Let \( u = f'(t) \) and \( dv = dt \). Then \( du = f''(t) \, dt \) and \( v = t - b \) (-b is chosen for convenience).

Then \( \int_a^b f'(t) \, dt = f'(t)(t-b) \bigg|_a^b - \int_a^b f''(t)(t-b) \, dt = f'(a)(b-a) + \int_a^b f''(t)(b-t) \, dt \)

Thus \( f(b) = f(a) + f'(a)(b-a) + \int_a^b f''(t)(b-t) \, dt \)

Now to evaluate this new integral, let \( u = f''(t) \) and \( dv = (b-t) \, dt \). Then \( du = f'''(t) \, dt \) and \( v = -\frac{(b-t)^2}{2} \). Then \( \int_a^b f''(t)(b-t) \, dt = f''(a)(b-a)^2 + \int_a^b f'''(t) \frac{(b-t)^2}{2} \, dt \)

Thus \( f(b) = f(a) + f''(a)(b-a) + f'''(a) \frac{(b-a)^2}{2} + \int_a^b f'''(t) \frac{(b-t)^2}{2} \, dt \)

This process can be continued indefinitely. However, you might, at this point, observe that the terms in the series have (increasing derivatives of \( f(x) \) at \( a \)) times (increasing powers of \( b-a \)) divided by (increasing factorials).

**Exercise:** Show that the next term in the series seen above, would be:
Chapter 9: Introduction to Calculus

Now, replacing \( b \) with \( x \) produces the power series formula which is called a **Taylor Series** after the Englishman Brook Taylor (1685-1731) who is given credit for its discovery.

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

Now when using a Taylor Series to approximate a function, you usually use some limited number of terms of the series. If you use

\[
\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k
\]

to approximate \( f(x) \), then the error, which is often called the remainder, is the next integral

\[
\int_a^b f^{(n+1)}(t) \frac{(b-t)^n}{n!} \, dt.
\]

Various computational forms for this remainder integral can be seen in a standard calculus text. We will not devote any further time to this work, however. It should be noted, however, that a special and simpler case of a Taylor Series can be obtained by letting \( a = 0 \) (expand about 0).

\[
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} (x)^k
\]

This series is called a **Maclaurin Series** after the Scotsman Colin Maclaurin (1698-1746) who is given credit for its discovery.

Recall that we saw at the beginning of this section that

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots
\]

by long division. Shown below is the creation of a Maclaurin Series for this function.
\( f(x) = (1-x)^{-1} \Rightarrow f(0) = 1 \)
\( f'(x) = (1-x)^{-2} \Rightarrow f'(0) = 1 \)
\( f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2 \)
\( f'''(x) = 6(1-x)^{-4} \Rightarrow f'''(0) = 3! \)

\[ \vdots \]
Thus \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \) which agrees with our previous result.

Now we know that this series can not be valid for all choices for \( x \). For example, as we saw before, if \( x = 2 \), then \(-1 = 1 + 2 + 4 + 8 + \cdots\), which is not true. Power Series are tested for convergence using the ratio test. The process is demonstrated below. Since the ratio test applies to positive valued series, absolute values are used.

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|. \]
Thus the series converges when \(|x| < 1\) and diverges when \(|x| > 1\). When \( x = 1 \), we have \( \sum_{k=0}^{\infty} 1 = \infty \), and hence the series diverges for \( x = 1 \). The series also diverges at \( x = -1 \), as we will explain in the next subsection. Thus the interval of convergence for \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \) is \( |x| < 1 \). So you see we would never choose 2 for \( x \), and thus the apparent problem of having a negative number equal an infinite sum of positive numbers is only an illusion.

**ALTERNATING SERIES**

For the series at the end of the last subsection, when \( x = -1 \), we have \( \sum_{k=0}^{\infty} (-1)^k \).

This is what is called an alternating series. It can be proven that an alternating series is convergent if it is strictly alternating and, in absolute value, the series is non increasing with the terms approaching 0. You have seen that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent. The alternating harmonic series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) is strictly alternating, non increasing in absolute value, and has terms approaching 0. Hence the alternating harmonic series converges.

**CREATING POWER SERIES**
Next, we compute the Maclaurin series for \( f(x) = e^x \). Since all the derivatives equal \( e^x \) and \( e^0 = 1 \), the series is easily seen to be \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

Now consider \( f(x) = \ln x \). Observe that \( f(x) \) and its derivatives do not exist at 0. Thus we can not express \( \ln x \) as a Maclaurin series and must use a Taylor Series (we will choose \( a = 1 \)).

\[
\begin{align*}
f(x) &= \ln x \quad \Rightarrow \quad f(1) = 0 \\
f'(x) &= x^{-1} \quad \Rightarrow \quad f'(1) = 1 \\
f''(x) &= -1(x^{-2}) \quad \Rightarrow \quad f''(1) = -1 \\
f'''(x) &= 2(x^{-3}) \quad \Rightarrow \quad f'''(1) = 2 \\
f^{(4)}(x) &= -6(x^{-4}) \quad \Rightarrow \quad f^{(4)}(1) = -6 \\
&\vdots \\
\end{align*}
\]

Thus \( \ln x = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k \)

**Exercise:** Show that the series seen above, is convergent on \((0,2]\).

Now, once a series is produced for a function, substitution can be used to produce a series for a related function. For example:

\[
\begin{align*}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}
\end{align*}
\]
Recall that the normal curve seen in statistics has an equation which is a modification of $e^{-x^2}$. To evaluate areas under this normal curve would necessitate integrating this modification of $e^{-x^2}$, which can't be done in the normal manner. By integrating the equivalent power series, an approximation for the desired area can be obtained. In this book we will not do any of this numerical work. Nor will we use series to solve DE's, a task for which series are well suited.

We will conclude this introduction to infinite series with a derivation by you, of the **COSMIC EQUATION** which relates the five most important constants in Mathematics.

---

**FINAL EXERCISE:**

First show that the Maclaurin Series for $\sin x$ and $\cos x$ are:

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

Second, using the series for $e^x$, $\sin x$ and $\cos x$; show that:

\[
e^{ix} = \cos x + i \sin x
\]

Third, let $x = \pi$ to obtain:

\[
e^{i\pi} + 1 = 0
\]

thus relating the five most important constants in mathematics $e, i, \pi, 1$ and 0.

---

**Problem Set 9.7**

1. For what values of $r$ is the sequence \{ $nr^n$ \} convergent?
Determine if each of the following converges or diverges [2 – 5].

2. \( \sum_{n=1}^{\infty} n^2 e^{-n^3} \)  
3. \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1} \)

4. \( \sum_{n=1}^{\infty} (-1)^n \frac{2}{n} \)  
5. \( \sum_{n=1}^{\infty} \frac{\tan \frac{1}{n}}{n} \)

6. Find the values for \( x \) for which \( \sum_{n=0}^{\infty} \frac{(x + 3)^n}{2^n} \) converges.

7. (a) Use the series for \( \frac{1}{1-x} \) seen earlier, to write the series for \( \frac{1}{1+x^2} \).

[Hint: Replace \( x \) by \(-x^2\).]

(b) Recall that \( \tan^{-1} x = \int \frac{1}{1+x^2} \, dx \). Replace the integrand by the series and integrate to find a series for \( \tan^{-1} x \).

(c) This series converges for \( |x| \leq 1 \). Substitute 1 for \( x \) to find a series of constants that sum to \( \pi \). This beautiful result is called the Leibniz formula for \( \pi \).
EPILOGUE

Despite the claim of using an inductive/deductive approach, quite often straight deduction was employed in this book. This borders on being Pure Mathematics. With regard to pure mathematics, Bertrand Russell (1872-1970) the English philosopher, logician and mathematician noted for his ground-breaking work in mathematical logic and the foundations of mathematics, once said:

Pure mathematics consists entirely of such assertions as that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true . . . If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics. Thus:

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Nevertheless, it is appropriate at the end of this book to observe that we started with the set of natural numbers and progressed through the sets of whole, integer, rational, real and finally complex numbers; incorporating set theory, logic, number theory, arithmetic, algebra, trigonometry and calculus in our study of elementary functions, their graphs, and applications. It is hoped that the journey was interesting and informative and that you really have succeeded in

Linking Concepts In Context - Mathematics For Teachers,
and that:

You Do Know What You Are Talking About
And What Is True When Teaching Mathematics,
because having covered 53 sections,
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