CHAPTER 8
ELEMENTARY FUNCTIONS II
TRIGONOMETRY

1. Trigonometric Functions
2. Triangles, Laws of Sines and Cosines
3. Trig Formulas and Equations
4. Parametric and Polar Equations
5. Vectors and Complex Numbers
8.1 TRIGONOMETRIC FUNCTIONS

Trigonometry is a branch of mathematics that was initially developed to solve problems involving triangles that occur in navigation and astronomy. The word trigonometry literally means *triangle measurement*. Today a student's first exposure to trig often occurs when studying similar triangles. Recall that similar triangles have the same shape but not necessarily the same size. This means that two similar triangles would have the same angle sets. Now since the sum of the angles for any triangle is 180°, if two triangles have two pair of equal angles then the third pair would also be equal.

**Note:** The angle measurement referenced here is degrees. You know that 1° is \(\frac{1}{360}\) of a complete revolution. This arbitrary measurement probably arose due to there being a few more than 360 days in a year and a number with many divisors being desired. Observe that one-half a revolution = 180°, one-third revolution = 120°, one-fourth revolution = 90°, one-fifth revolution = 72°, one-sixth revolution = 60°, one-eighth revolution = 45°, one-ninth revolution = 40°, etc. Later you will work in another measurement scheme called radians.

An important feature of similar triangles is that for any set of similar triangles, the ratios of pairs of corresponding sides are equal. This feature is pictured below.

Later in this chapter, applications involving solving triangles will be examined. Initially, in the triangle applications, right triangle problems will be studied. Consider the right triangle shown below.

The side opposite the right angle is called the **hypotenuse**. The side opposite angle \(\theta\) is called the **opposite side** and the side adjacent to angle \(\theta\) is called the **adjacent side**. There are six different ratios of the sides for this triangle. Using the first letter of the names of the sides these ratios are \(\frac{o}{h}, \frac{a}{h}, \frac{o}{a}\) and the reciprocals of these three ratios. Any two right triangles having equal angles \(\theta\) will be similar by AA Similarity, and hence have these corresponding ratios equal. Consequently, if we were to make a table listing these ratios for various measures for \(\theta\), we could use the table in solving right triangle problems. The example below illustrates the technique.
**Example:**

\[
L \quad 30^\circ \quad 10 \text{ m}
\]

Determine Length L

Solution: As we will develop later, the ratio \( \frac{o}{h} \) for \( \theta = 30^\circ \) is \( \frac{1}{2} \).

Thus \( \frac{10 \text{ m}}{L} = \frac{1}{2} \) and hence \( L = 20 \text{ m} \).

From the preceding discussion, you can see that the ratio values depend on the angle \( \theta \) measurements. In other words a particular set of ratios is a function of \( \theta \). At this time let us say that \( \frac{o}{h} = s(\theta) ; \frac{a}{h} = c(\theta) ; \) and \( \frac{o}{a} = t(\theta) \). These three functions along with their reciprocal functions generate ratios of sides of a measured triangle and are consequently referred to as trigonometric functions. Now each ratio actually consists of two variables. Thus we have the somewhat awkward situation of dealing with functions with one independent variable but two dependent variables. We also need better names than \( s, c, \) and \( t \) for our functions. We will deal with these deficiencies in the next subsection.

**THE UNIT CIRCLE**

Imagine a line segment, \( \textbf{OP} \), of length 1 with endpoint \( \textbf{O} \) fixed at the origin, which rotates counterclockwise through an angle \( \theta \). The other endpoint, \( \textbf{P} \), will sweep out the circle shown below.

Observe that when \( \theta = 0 \), \( \textbf{P} = \textbf{A} \); when \( \theta = 90^\circ \), \( \textbf{P} = \textbf{B} \); when \( \theta = 180^\circ \), \( \textbf{P} = \textbf{C} \); when \( \theta = 270^\circ \), \( \textbf{P} = \textbf{D} \); when \( \theta = 360^\circ \), \( \textbf{P} = \textbf{A} \) again; when \( \theta = 450^\circ \), \( \textbf{P} = \textbf{B} \) again; etc. You should also realize that if \( \textbf{OP} \) rotates clockwise (in which case \( \theta \) is said to be negative), \( \textbf{P} \) would trace out the same circle. Now drop a perpendicular from \( \textbf{P} \) to the x-axis, forming
a right triangle. The hypotenuse for this right triangle has 1 unit as its length. Label the horizontal side (OE) \(x\) and the vertical side (PE) \(y\). Then the coordinates for \(P\) are \((x,y)\). Now since \(P(x,y)\) depends on \(\theta\), \((x,y)\) is a function of \(\theta\) and we write \((x,y) = f(\theta)\). Now we have situation similar to that seen in the previous subsection, i.e. a function with one independent variable and two dependent variables. Actually for \(0^\circ < \theta < 90^\circ\), we have the very same situation because \(\frac{h}{a} = y; \frac{a}{h} = x\); and \(\frac{a}{h} = \frac{y}{x}\).

Now observe that when \(P\) is in a quadrant other than the First Quadrant and a perpendicular is dropped to the \(x\)-axis forming a right triangle, then \(x\) or \(y\) or both will be negative. So we now have \((x,y) = f(\theta)\) where \(\theta\) is a positive, zero or negative number and likewise \(x\) and \(y\) are positive, zero or negative numbers. To eliminate the awkward situation of having two dependent variables, begin by creating two new functions temporarily called \(f_1\) and \(f_2\) such that \(y = f_1(\theta)\) and \(x = f_2(\theta)\). Now what should be the permanent names for these functions? Look back at the unit circle drawing. If the perpendicular dropped to the \(x\)-axis from \(P\) were extended until it met the circle again at a point \(F\), Chord \(PF\) would be formed. Chord \(PF\) could be though of as a bowstring for bow arc \(PF\). Then \(PE\) would represent half a bowstring. The Latin word \(Sine\) was intended to be a translation for the Arabic word for \(half \ a \ bowstring\), but a mistake was made in the translation. Sine actually means half a gulf or bosom. Nevertheless the name stuck, and today we write the function \(y = \text{sine } \theta\) which is usually abbreviated as \(\sin \theta\). The other new function, \(f_2\) is named cosine and abbreviated as \(\cos\); thus \(x = \cos \theta\). The reason for the name cosine comes from geometry. Examine Right Triangle \(ABC\) below.

![Right Triangle ABC](image)

\[
\sin A = \frac{a}{c} \quad \sin B = \frac{b}{c}
\]

Since \(C = 90^\circ\), \(A + B = 90^\circ\) and \(A\) and \(B\) are complementary angles. Thus the \(\sin B = \sin \text{ (compliment of } A) = \cosine A = \cos A = \frac{b}{c}\).

So for our unit circle, \(x = \cos \theta\) and \(y = \sin \theta\).

Now there are four other trig functions which are defined below.

\[
\begin{align*}
\text{tangent } \theta &= \tan \theta = \frac{y}{x} \\
\text{cotangent } \theta &= \cot \theta = \frac{x}{y} \\
\text{secant } \theta &= \sec \theta = \frac{1}{x} \\
\text{cosecant } \theta &= \csc \theta = \frac{1}{y}
\end{align*}
\]

All six trig functions can be pictured in one diagram as seen below.
Before leaving this subsection, several relationships between the trig functions should be noted and, as an exercise by you, confirmed.

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}
\]

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

**Exercise:** Confirm the seven statements seen above.

**Radian Measure**

Prior to this point, we have been measuring angles in degrees. In more advanced mathematics courses, **Radians** are employed. Examine the picture below.

Central angle \( \theta \) intercepts an arc of length \( s \) on a circle with radius \( r \).

If \( s = r \), then \( \theta \) is said to have measure 1 radian.
Now recall that the perimeter (which is called its circumference) of a circle with radius \( r \) is given by \( C = 2\pi r \) (\( \pi \) is irrational and approximately 3.14159 to 5 places). Hence there are \( 2\pi \) radians in a complete circle. Thus \( 2\pi \) radians \( = 360^\circ \); \( \pi \) radians \( = 180^\circ \); \( \frac{\pi}{2} \) radians \( = 90^\circ \); \( \frac{\pi}{3} \) radians \( = 60^\circ \); etc. In future work it will, on occasion, be necessary to convert between radians and degrees. Since \( \pi \) radians \( = 180^\circ \), \( \frac{\pi R}{180^\circ} = 1 \). Thus to convert from degrees to radians, multiply by \( \frac{\pi R}{180^\circ} \) and to convert from radians to degrees, divide by \( \frac{\pi R}{180^\circ} \). As examples \( 30^\circ = \frac{\pi R}{180^\circ} \), \( 30^\circ = \frac{\pi}{6} \) radians and \( \frac{2\pi}{3} \) radians \( = \frac{180^\circ}{\pi R} \frac{2\pi}{3} R = 120^\circ \).

One radian would equal \( \frac{180^\circ}{\pi} \) or about 57.3^\circ.

Radian measure, when used in conjunction with the unit circle discussed in the previous subsection, facilitates what is frequently called trig functions of real numbers. Look back at the picture used to define radian measure. If \( \theta \) is one radian and \( r = 1 \) (as on a unit circle) then \( s = 1 \). Simply put, on a unit circle, the measure of a central angle in radians is the same number as the length of the intercepted arc! This prompts the notion of measuring arcs on a circle in the same way as measuring the central angle that intercepts the arc. Note that radians or degrees could be used. For example a \( 60^\circ \) arc would be one-sixth of a circle.

Finally, since most of our future work will involve using radian measure, the "R exponent" will no longer be used. Unless degree measure is indicated the measure will be understood to be radian. For example evaluate \( \sin 2 \) means evaluate the sine of \( 2 \) radians. Evaluating trig function values will be discussed in the next subsection.

EVALUATING TRIG FUNCTION VALUES

From ancient times up to the advent of calculators and computers, trig function values were found using tables. Two thousand years ago, tables of chord lengths for arcs from \( \frac{1}{2}^\circ \) to \( 180^\circ \) in \( \frac{1}{2}^\circ \) increments on a circle of radius 60 were used in astronomy and in solving right triangles. Eventually it was realized that using tables for half-chord lengths would simplify computation. These half-chord lengths tables led to the modern trig tables still seen today. Of course very few people use tables now, since using calculators and computers is easier. Calculators and computers are programmed to provide decimal approximations for \( \sin \theta \), \( \cos \theta \), and \( \tan \theta \) where \( \theta \) is measured in radians or degrees (a setting must be selected).

Note: To obtain \( \csc \theta \), \( \sec \theta \), and \( \cot \theta \) values simply use the reciprocal key (1/x) along with \( \sin \), \( \cos \), and \( \tan \) respectively on your calculator.

Now the exact trig function values for arcs that terminate on an axis are easy to find. For example, \( \sin 90^\circ = \sin \frac{\pi}{2} = 1 \) and \( \tan (- 180^\circ) = \tan (- \pi) = 0 \). Other exact trig function values can be found using some geometry. First of all, consider the isosceles...
right triangle ABC seen on the left below. Observe that \( \theta = 45^\circ = \frac{\pi}{4} \) and thus \( \sin 45^\circ = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \). Similarly, \( \cos 45^\circ = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \). The other four trig function values for this arc (or angle) can also be found using the picture or by employing their relationship to \( \sin \) and \( \cos \).

To find some other exact trig function values, examine Triangle DEF seen above and work through the exercise below.

**Exercise:** 1. What type of triangle is DEF?
2. How is G related to D and F?
3. Thus what is EG?
4. Why is EG also an altitude and an angle bisector?
5. What kind of triangle is DEG?
6. What are the measures of \( \alpha \) and \( \theta \)?
7. Write the six trig function values for each of these two angles.

Now that you know how to find exact trig function values for the special angles of \( 30^\circ, 45^\circ \) and \( 60^\circ \) or their radian counterparts of \( \frac{\pi}{6}, \frac{\pi}{4} \) and \( \frac{\pi}{3} \), you should realize that you can also find exact trig function values for any angles, positive or negative, that produce right triangles in the plane with those special angles. For example suppose that \( \theta = \frac{7\pi}{6} \, (210^\circ) \) and work through the exercise below.
Exercise: The angle is drawn below.

![Angle Diagram]

What is the measure of this angle?

Determine x and y (remember they are negative).

Now write the six trig function values for 210°.

GRAPHS OF THE TRIG FUNCTIONS

Now that you know some trig function values it is time to look into graphing trig functions. We will begin with $y = \sin x$. Notice that we are employing the familiar scheme with x as the independent variable and y as the dependent variable. This is perfectly reasonable since we now know that a trig function of a permitted domain value real number generates a range value real number. If we make an x-y table with the sine values we know or could find, plot those points and realize that a sine graph is a smooth curve (continuous with no breaks or corners or cusps), we produce the graph shown below.

Observe that the graph cycles between -1 and 1 and repeats a cycle every $2\pi$ units. Functions whose graphs exhibit such behavior are called **Periodic Functions**. For the standard Sine Function the period is $2\pi$. Also observe that the graph is symmetric about the origin and thus $y = \sin x$ is an odd function and $\sin (-x) = -\sin x$.

The graph of $y = \cos x$ can be produced in the same manner and is shown below.
Observe that this graph also cycles between -1 and 1 and repeats a cycle every $2\pi$ units. $y = \cos x$ is also a Periodic Function with period $2\pi$. For this function observe that the graph is symmetric about the y-axis and thus $y = \cos x$ is an even function and $\cos (-x) = \cos x$.

Likewise the graph of $y = \tan x$ can be produced and is shown below.

As you can see this graph is significantly different from that for the sine and cosine functions. It is periodic, but its period is only $\pi$. An even more significant difference is that this graph has asymptotes (at odd multiples of $\pi/2$). Each of the other trig functions will also have asymptotes.

Since the function $\csc x$ is the reciprocal of $\sin x$, it is easier to visualize the graph of $y = \csc x$ together with the sine graph as shown below.

Likewise $y = \cos x$ and its reciprocal $y = \sec x$ are shown below.
Finally, \( y = \tan x \) and its reciprocal \( y = \cot x \) are shown below.

This last set of pictures is somewhat confusing so the asymptotes are not displayed. Some thought and reviewing the patterns seen in the other graph pairings should lead you to the realization that for the reciprocal graphs pairs, where one is zero the other has an asymptote and also that the graphs intersect where the values are one.

**Exercise:** You saw that sine is an odd function and cosine is an even function. Determine, for each of the other four trig functions, if it is even, odd or neither.

Now that the basic trig graphs have been examined, it is a fairly easy matter to graph **transformed** versions of these graphs. The transformation rules (shown below) are reviewed once again.

Let \( y = f(x) \) and let \( a \) and \( b \) be positive numbers.
A general Sine Function takes the form \( y = d + a \sin (bx + c) \).
Let \( a = -3 \), \( b = 2 \), \( c = -\pi/2 \), and \( d = 2 \), producing the equation \( y = 2 - 3 \sin (2x - \pi/2) \)
To graph this equation first rewrite it as \( y = 2 - 3 \sin [2(x - \pi/4)] \).
The transformations on the standard sine graph are flip over the x-axis, horizontal shrink by 2, vertical stretch by 3, right \( \pi/4 \), and up 2. The graph is shown below.

Now with trig functions the transformations have special names.
First, you have seen that the trig functions are periodic which means that they repeat a particular cycle, and that the **period** for a standard sine graph is \( 2\pi \). For the example above, the graph is horizontally shrunk by a factor of 2. Thus the period = \( \pi \). Observe that this period value is the standard period value divided by the coefficient of \( x \) which in our example is equal to 2. Thus the formula for the period in terms of our general sine equation is \( \text{Period} = \frac{2\pi}{|b|} \). **Note:** The "b" is in absolute values because period is a distance.

Second, you have seen that a standard sine wave cycles between -1 and 1 for a range of 2 units. The **amplitude** of a wave is defined as half its range. In our example the range is 6 so the amplitude is 3. Observe that for our general sine equation \( \text{Amplitude} = |a| \).
Third, a standard sine wave spirals about the x-axis. The wave in our example is translated up 2 units and spirals about \( y = 2 \). This vertical translation is called **vertical displacement**. For our general sine equation \( \text{Vert. Disp.} = d \).

Finally, for a standard sine wave, when \( x = 0 \), \( y = 0 \), the **Primary Cycle** is said to start at 0. For the example we have been working with, the graph is translated right \( \frac{\pi}{4} \) units. Thus the primary cycle for our example's graph starts at \( \frac{\pi}{4} \). The starting point for a graph's primary cycle is called its **Phase Shift**. For the general sine wave equation, \( \text{Phase Shift} = -\frac{c}{b} \).

### Summary:

For a general sine function \( y = d + a \sin (bx + c) \)

- **Period** = \( \frac{2\pi}{|b|} \)
- **Amplitude** = \( |a| \)
- **Vert. Disp.** = \( d \)
- **Phase Shift** = \( -\frac{c}{b} \)

The terminology related to transformations on a general sine function apply with some modification to the other five trig functions. No modification is required for the cosine function. The period formula for the tangent and cotangent functions is \( \text{Period} = \frac{\pi}{|b|} \) since these function's graphs cycle in \( \pi \) units. Finally only the sine and cosine functions' graphs have amplitude (because the other four all go to infinity).

In the preceding discussion, you saw that the Primary Cycle for the standard trig graphs start at \( x = 0 \). This fact prompts an alternative procedure for finding the phase shift. Simply set the argument \( bx + c \) equal to 0, producing \( x = -\frac{b}{c} \). Note that you could continue this technique by setting the argument equal to the standard period (2\( \pi \) or \( \pi \)) and thus find where the primary cycle ends.

### The Inverse Trig Functions

Consider the equation \( \sin x = \frac{1}{2} \). To solve for \( x \), we must find the angle (or arc) that you would take the sine of to get \( \frac{1}{2} \). We can denote this by writing \( x = \arcsine \frac{1}{2} \). One of the answers would be \( \frac{\pi}{6} \) or 30°. To generalize this notation if \( y = \sin x \), then \( x = \arcsine y \).

Now recall that when finding the inverse of a relation \( y = f(x) \), you interchanged \( x \) and \( y \) and then solved for \( y \). If we do this with \( x = \arcsine y \), we obtain \( y = \arcsine x \) which would then be the inverse of \( y = \sin x \). Now if we borrow the notation used previously for inverse functions, i.e. the inverse of \( y = f(x) \) is \( y = f^{-1}(x) \), then another notation for the inverse of \( y = \sin x \) is \( y = \sin^{-1} x \).
In the last subsection you saw the graph of \( y = \sin x \) to spiral about the x-axis. Since the graphs of inverses are symmetric about the line \( y = x \), the graph of \( y = \sin^{-1} x \) will spiral about the y-axis as shown below.

![Graph of \( y = \sin^{-1} x \)](image)

Clearly, by looking at the graph, you see that \( y = \sin^{-1} x \) is not a function, but we can create a function by restricting the range. When working with the inverse sin function it is customary (but not essential) to understand that the range is restricted to \([-\pi/2, \pi/2]\).

**Note:** This feature is built in to calculators and computers.

For example for \( \sin^{-1} (-1/2) \), when in degree mode, a calculator will return -30°.

The preceding discussion, with some modifications, applies to all the trig functions, so each of them has a corresponding inverse function. You will look at \( y = \cos^{-1} x \) and \( y = \tan^{-1} x \) in the problem set that follows. Inverse trig functions are used extensively in calculus.

This section has served to introduce the trig functions and their graphs. Trigonometry continues to have great utility in solving a large collection of problems across many disciplines. You will see some of the applications of trigonometry in the next chapter.

**Problem Set 8.1**

1. Convert to exact radian measure.

   (a) -120° (b) 900°

2. Convert to exact degree measure.
3. Determine the exact values.

(a) \( \sin \frac{7\pi}{6} \)  
(b) \( \cos \frac{5\pi}{3} \)  
(c) \( \tan 135^\circ \)  
(d) \( \sec 600^\circ \)

4. State the phase shift, vertical displacement, period and amplitude (if it exists) and sketch the graph.

(a) \( y = -3 + 2 \sin(3x - \pi) \)  
(b) \( y = 2 - \cos(2x + \frac{\pi}{2}) \)

(c) \( y = -2 - 3 \tan \frac{1}{2} x \)  
(d) \( y = 2 + 3 \csc (2x - \pi) \)

5. Find my sine function equation using the following clues.

A. My period is \( 6\pi \).
B. My amplitude is 3.
C. My phase shift is \(-\frac{\pi}{2}\).
D. My vertical displacement is +4.

6. A police car is parked 10 feet East of a point \( Q \) on a long straight wall running North and South. The beacon on top of the car is rotating so that a light spot on the wall is at a distance \( d \), from point \( Q \) after \( t \) seconds, where \( d \) is given by the equation

\[ d(t) = 10 \tan (2\pi t) \]

When \( d \) is positive, the light spot is north of point \( Q \); when \( d \) is negative, the light spot is south of point \( Q \).

(a) Graph the function on the interval \([0,2]\).
(b) Explain the meaning of the values of \( t \) for which \( d(t) \) is undefined.

7. The tidal motion of oceans results in water levels along shores rising and falling periodically. Water heights are measured as the amounts above or below the mean low water height. At a certain location the high tide on a beach occurred at 3:38 am, and the low tide occurred at 9:53 am. The height of the water at high tide was 9.3 feet and at low tide, the height of the water was −0.7 feet.

(a) When will the next high tide occur on the beach?
(b) Find a sine wave function that fits the data.
(c) Using your function, find the height of the water at 6 pm.
8. Evaluate:

(a) $\sin^{-1} -\frac{1}{2}$  
(b) $\cos^{-1} 1$  
(c) $\sin \left( \cos^{-1} \frac{1}{2} \right)$

(d) $\sin \left( \sin^{-1} \frac{1}{2} \right)$  
(e) $\cos^{-1} \left( \sin -\frac{\pi}{6} \right)$

9. Simplify:

(a) $\sin \left( \sin^{-1} x \right)$  
(b) $\sin \left( \cos^{-1} x \right)$

10. Graph the inverse trig functions.

(a) $y = \frac{\pi}{6} + \sin^{-1} 2x$  
(b) $y = -\frac{\pi}{3} + \cos^{-1} \frac{1}{2} x$
8.2 TRIANGLES, LAWS OF SINES AND COSINES

In the last chapter you were introduced to right triangle problems. In this section you will see how to solve triangle problems if the triangle is right or oblique. A triangle has six parts - three side and three angle measurements. When solving a triangle problem, generally you are given some measurements and asked to find other measurements for the triangle. Now it is important to know when a given set of measures would produce exactly one triangle, more than one triangle, or possibly, no triangle. You learned in geometry that given three side lengths, for which no one side is longer than the sum of the other two, a unique triangle can be constructed. This fact is used to prove triangles congruent and is frequently called the SSS Congruency Postulate. You likewise learned that if given two sides and the included angle or two angles and the included side, a unique triangle can be constructed. These facts are also used to prove triangles congruent and are frequently called the SAS and ASA Congruency Postulates respectively.

Now when solving right triangle problems, in addition to the right angle, you are either given a side and one of the other angles or given two sides. Thus you are essentially given or can produce SSS, SAS or ASA, and can solve this unique triangle for its other parts. For example, consider the right triangle shown below.

![Right Triangle Diagram]

You know $C = 90^\circ$. Given $a$ and $b$, you have SAS. If you are given $a$ and $c$, then you can use the Pythagorean Theorem to find $b$, and you have SAS again. If you are given $a$ and Angle $B$, then you have ASA. If you are given $b$ and Angle $B$, then you have ASA because knowing two angles of a triangle tells you the third angle since the angle sum for a triangle is $180^\circ$.

**Question:** Why could you not solve Triangle ABC seen above if only given Angles A and B?

A typical problem and its solution is shown below.

A ladder 10 feet long is leaning against a wall at a $70^\circ$ angle. How far up the wall does the top of the ladder reach?

\[
\sin 70^\circ = \frac{h}{10}
\]

\[
h = 10 \sin 70^\circ - 9.4 \text{ feet}
\]
In order to solve non-right triangles, which are usually called oblique triangles, we need some additional tools called the **Law of Sines** and the **Law of Cosines**.

**THE LAW OF SINES**

To derive the Law of Sines, consider the triangle drawn below.

![Image of a triangle with sides labeled a, b, c and angles labeled A, B, C]

\[
\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a} 
\]

Hence \( h = b \sin A = a \sin B \) and thus \( \frac{\sin A}{a} = \frac{\sin B}{b} \)

Similarly we could show \( \frac{\sin B}{b} = \frac{\sin C}{c} \) and so by transitivity:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

In words, the Law of Sines says: for any triangle the ratio of the sine of an angle to its opposite side is constant.

*Note:* The reciprocals of these ratios are also equal.

Now observe that the Law of Sines is a proportion relating angles and opposite sides. If you know two angles and a side opposite one of the known angles, you can use the proportion to find the side opposite the other known angle.

*Note:* Recall that AAS is equivalent to ASA and hence there is exactly one triangle with two angle and one side measurements.

The example below illustrates the process.

\[
\frac{x}{\sin 80^\circ} = \frac{20}{\sin 50^\circ} \quad \Rightarrow \quad x = \frac{20 \sin 80^\circ}{\sin 50^\circ} \approx 25.7
\]
Now, if you know two sides and an angle opposite one of the known sides, you might expect that you could use the proportion to find the angle opposite the other known side. However, recall that SSA is not a triangle congruence postulate. If you are given a data set which consists of the measures for two sides and an angle that is to be opposite one of the given sides, then there are 0, 1 or 2 possible triangles that can be constructed. To understand when there are no solutions examine the two pictures below. In the figure on the left, angle A is acute and for the picture on the right, angle A is obtuse. In both cases side \( a \) is not long enough to create a triangle.

Happily, when given such a data set, with which a triangle could not be constructed, if you try to use the Law of Sines to solve for Angle B, you will obtain an equation such as \( \sin B = 1.3 \) which is impossible (sine values lie between -1 and +1) and you will know there is no solution. Now if the given angle is obtuse and there is a solution, there is only one solution. But if the given angle is acute and there is a solution, there could be a second solution. It is for this reason that SSA is often referred to as the Ambiguous Case. Consider the pictures below.

If \( a \) is just long enough to swing down and touch the base line (when \( a = b \sin A \)), one right triangle is formed. If \( a \) is as long or longer than \( b \) one triangle is formed. However if \( a \) is between \( b \sin A \) and \( b \) then there are two possible triangles and hence, when using the Law of Sines, you must find both possible measures for angle B. An example is shown below.

**Example:** Solve the triangle for which \( A = 30^\circ \), \( b = 4 \) meters and \( a = 3 \) meters.

\[ b \sin A = (4 \text{ meters}) \sin 30^\circ = 2 \text{ meters} \]

Thus \( a = 3 \) meters is between \( b \sin A \) and \( b \) and there are two solutions for B.

Using the Law of Sines, \( \frac{\sin B}{4 \text{ m}} = \frac{\sin 30^\circ}{3 \text{ m}} \)

and \( \sin B = \frac{4 \cdot \frac{1}{2}}{3} = \frac{2}{3} \)

Thus \( B = 41.8^\circ \) or \( 180^\circ - 41.8^\circ = 138.2^\circ \).

Now as an exercise, you might determine the possible corresponding values for Angle C and Side c in the example above.
THE LAW OF COSINES

To derive the Law of Cosines, consider the triangle drawn below.

\[ a^2 = h^2 + (c - x)^2 = h^2 + c^2 - 2cx + x^2 \]

but \[ b^2 = h^2 + x^2 \]

thus \[ a^2 = b^2 + c^2 - 2cx \]

Now \[ \cos A = \frac{x}{b} \]

\[ a^2 = b^2 + c^2 - 2bc \cos A \]

In words, the square of one side of a triangle equals the sum of the squares of the other two sides minus twice the product of those two sides times the cosine of the angle included between those two sides.

Observe that the Law of Cosines relates all three sides of a triangle with one of the angles. Consequently it can be used when given all three sides or two sides and an angle. However the Law of Sines is easier to use, so generally we use it with angles and opposite sides (ASA or the Ambiguous Case SSA). Thus you should use the Law of Cosines only with SSS or SAS. An example (with an exercise) follows.

**Example:** Solve the triangle for which \( a = 2, b = 3 \) and \( c = 4 \).

\[ 4^2 = 3^2 + 2^2 - 2(3)(2) \cos C \]

\[ 16 = 9 + 4 - 12 \cos C \]

\[ 12 \cos C = -3 \]

\[ \cos C = -\frac{1}{4} \]

\[ C = 104.5^\circ \]

**Exercise:** Finish the example above by finding the other angles of the triangle. Use the Law of Cosines and/or the Law of Sines.
AREA OF A TRIANGLE

In this section we have been dealing with both right and oblique triangles. We conclude this section with another application of trig, which is finding the area of a triangle. Although the standard \( A = \frac{1}{2} b h \) formula, for a triangle's area in terms of its base and height, can frequently be applied, it is often the case that we only know SSS, SAS or ASA about some triangle. If we know ASA or SSS we can apply the Law of Sines or Cosines to get to SAS. Consequently what would be most useful is a SAS Trig Formula for the area of a triangle. This formula is developed below.

\[ \sin A = \frac{h}{c} \]
\[ h = c \sin A \]
\[ \text{Area} = \frac{1}{2} \cdot b \cdot h = \frac{1}{2}bc \sin A \]

In words, the area of the triangle equals one-half the product of two sides times the sine of the angle included between the two sides.

Problem Set 8.2

1. A Naval aviator is about to land her Tomcat on the deck of a carrier. If she is at an altitude of 8 miles and 50 miles out (horizontal distance) at what depression angle must she land? How long will it take her to reach the carrier if she flies at 600 mph?

2. In 1981 I hiked to the bottom (and back to the top in one long day) of the Grand Canyon from the South Rim which is about 5000 feet above the Colorado river at the bottom of the canyon. A few years later, from a point on the North Rim of the canyon, I observed the angle of depression to the South Rim to be 1.3°. I was told that the horizontal distance between the two rims is about 10 miles. How many more feet than from the South Rim, would I have to descend in order to hike to the bottom from the North Rim? [Note: From the South Rim, I hiked the Bright Angel Trail which is about 8 miles to the bottom, whereas from the North Rim the trail is about 12 miles to the bottom.]

3. The Great Pyramid of Cheops in Egypt has a square base 230 meters on each side. The faces of the pyramid make an angle of 51.833° with the horizontal.
   (a) How tall is the pyramid?
   (b) What is the shortest distance you would have to climb up a face to reach the top?
   (c) Show that the ratio of the answer from part b to half the length of the base is very close to the Golden Ratio which is \( \frac{1 + \sqrt{5}}{2} \).
4. Solve the following problem using right triangle trig. From a point on a level stretch of road, at an elevation of 5000 feet, going due West towards the Front Range of the Rockies, you measure the angle of elevation to the top of a peak to be $20^\circ$. From a point $\frac{1}{2}$ mile closer to the Front Range, the angle of elevation to the same peak is $22^\circ$. What is the altitude of the peak?

5. Solve problem 4 using oblique triangle trig.

6. In the figure shown below, a boy scout camp is at point B and a girl scout camp is at point G. What is the straight line distance between the camps?

```
\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) -- (2,0) node[midway,above] {2 mi} -- (2,2) node[midway,left] {$120^\circ$} -- (0,2) node[midway,below] {3 mi} -- (0,0);
\end{tikzpicture}
\end{figure}
```

7. In problem 6 above, line PG represents a long straight road. Suppose a scout store is to be located on PG at a point S so that BS = GS. How far from P should this store be located?

8. In the figure shown below, an artillery cannon is located at Point A. A convoy $\frac{1}{2}$ mile long is proceeding along the road from R towards P at 30 mph. If the cannon has a range of 2 miles, how long will there be viable targets?

```
\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) -- (2,0) node[midway,above] {3 mi} -- (2,3) node[midway,right] {$30^\circ$} -- (0,3) -- (0,0);
\end{tikzpicture}
\end{figure}
```

9. I own a small triangular piece of property in Manhattan where the average value of land is $827,000 per acre. My property measures 80 ft, 90 ft and 120 ft on its three sides. My surveyor failed to give me any angle measures. Find the angles of my land and then find how much its worth. [1 acre = 43,560 sq. ft.]

10. Heron's Formula will calculate the area of a triangle, given only the lengths of its sides. Derive this formula, which is stated below, and then use it to determine the area of the triangular piece of land in problem 9 above.

Let $s = \frac{a + b + c}{2}$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
8.3 TRIG FORMULAS AND EQUATIONS

In the last section you worked with several trig formulas applicable to triangle problems. In this section you will develop more trig formulas applicable in a variety of problems.

SECTOR ARC LENGTHS AND AREAS, ANGULAR VELOCITY

Reexamine the picture previously used to define radian measure (shown again below).

\[
\frac{\theta}{2\pi} = \frac{s}{2\pi r} \Rightarrow s = r\theta \quad \text{where } \theta \text{ is measured in radians.}
\]

In a similar manner, an area of a sector formula can be developed as shown below.

\[
\frac{\theta}{2\pi} = \frac{\text{Area}_{\text{sector}}}{\pi r^2} \Rightarrow \text{Area}_{\text{sector}} = \frac{1}{2} r^2 \theta \quad \text{where } \theta \text{ is measured in radians.}
\]

Now imagine this circle to be a wheel rolling along the ground. We can describe the rate at which the wheel turns, which is called angular velocity, either in revolutions per time unit (such as rpm which is revolutions per minute) or by indicating a rate at which the angle changes in degrees or radians per time unit. Now observe that the relationship between these two methods of indicating angular velocity is simple.

One revolution per time unit = 2\pi radians (or 360°) per time unit.

A natural question that might occur at this time is, "How long would it take the wheel to roll a certain distance?" One way to answer this question involves converting angular velocity to linear velocity. Linear velocity is the rate of change in distance with respect to change in time (such as miles per hour - mph).

Consider a unit circle "wheel" rolling on a road as shown below.
The distance the wheel rolls is a function of its number of revolutions. Since this wheel has a circumference of $2\pi$ feet, in one complete revolution the wheel rolls $2\pi$ feet. So if our wheel is turning at 5 rpm ($10\pi$ radians per minute) it is rolling at $5 \times 2\pi = 10\pi$ feet per minute.

**Note:** Observe that for a unit circle the angular velocity and the linear velocity are represented using the same number!

Now to generalize the concept of angular velocity, we use the arc length formula, $s = r\theta$ where $\theta$ is measured in radians. The rate of change in $s$ is the linear velocity and is denoted by $v$. The rate of change in $\theta$ is the angular velocity and is usually denoted by $\omega$. Thus:

$$v = r\omega$$

In words, the linear velocity equals the radius of rotation times the angular velocity (in radians/time unit)

**Example:** A satellite in a circular orbit 900 km above the surface of the Earth makes a complete revolution every 2 hours. What is its linear velocity? Use 6400 km for the Earth's radius.

$$\omega = \frac{\pi}{2} \text{ rad/hour} \Rightarrow v = 7300 \pi \text{ kph}$$

**IDENTITIES**

In the next subsection you will be working with trig equations. The solution of many trig equations requires applying some trig relationships known as **identities**. An identity is an equation which is valid for all permissible values of the variable. For example the linear equation $x + 2 = \frac{1}{2}(2x + 4)$ is an identity. Now you have already seen several examples of trig identities, some of which are listed below.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}$$

$$\sin(90^\circ - \theta) = \cos \theta \quad \tan(90^\circ - \theta) = \cot \theta \quad \sec(90^\circ - \theta) = \csc \theta$$

$$\sin (- \theta) = -\sin \theta \quad \cos (- \theta) = \cos \theta \quad \tan (- \theta) = -\tan \theta$$

$$\csc (- \theta) = -\csc \theta \quad \sec (- \theta) = \sec \theta \quad \cot (- \theta) = -\cot \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

At this point some more complicated identities will be developed. Shown below is a triangle with the altitude drawn to one of the sides.
Looking at the whole triangle and applying the Law of Sines, we have

\[
\frac{\sin(x + y)}{e + f} = \frac{\sin (90^\circ - y)}{b}
\]

thus

\[
\frac{\sin(x + y)}{e + f} = \frac{h}{ab}
\]

and

\[
\sin (x + y) = \frac{h(e + f)}{ab} = \frac{he}{ab} + \frac{hf}{ab} = \frac{e}{b} \cdot \frac{h}{a} + \frac{h}{b} \cdot \frac{f}{a}
\]

so we have \( \sin (x + y) = \sin x \cos y + \cos x \sin y \)

**The sine of a sum equals the sine of the first times the cosine of the second plus the cosine of the first times the sine of the second.**

A comparable identity for the sine of a difference \( \sin (x - y) \), can easily be developed using the notions that the sine function is odd and the cosine function is even, as shown below.

\[
\sin (x - y) = \sin (x + (-y)) = \sin x \cos (-y) + \cos x \sin (-y)
\]

\[
= \sin x \cos y - \cos x \sin y
\]

Thus

\[
\sin (x - y) = \sin x \cos y - \cos x \sin y
\]

**The sine of a difference equals the sine of the first times the cosine of the second minus the cosine of the first times the sine of the second.**

Sometimes you see these two sine formula written as one formula as

\[
\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y
\]

(use the top symbol on each side or the bottom symbol on each side)

Now the cosine of a sum could be developed using triangles in a manner similar to that seen for the sine of a sum, but it is easier to use cofunction relationships.

\[
\cos (x + y) = \sin [90^\circ - (x + y)] = \sin [(90^\circ - x) - y]
\]

\[
= \sin(90^\circ - x) \cos y - \cos(90^\circ - x) \sin y
\]

Thus

\[
\cos (x + y) = \cos x \cos y - \sin x \sin y
\]

**The cosine of a sum equals the cosine of the first times the cosine of the second minus the sine of the first times the sine of the second.**

**Exercise A:** Show that the cosine of a difference formula is given by;
\[ \cos(x - y) = \cos x \cos y + \sin x \sin y \] and write what it says in words.

**Exercise B:** State the compact form for the cosine sum and difference formulas,

Next, we derive the tangent sum formula using the relationship \( \tan x = \frac{\sin x}{\cos x} \).

\[ \tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} \]

Dividing all the terms on the right side by \( \cos x \cos y \) produces

\[ \tan(x + y) = \frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y} \]

and thus

\[ \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \]

The tangent of a sum equals the tangent of the first plus the tangent of the second all divided by one minus the product of the tangent of the first times the tangent of the second.

**Exercise A:** Show that \[ \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \] and write what it says in words.

**Exercise B:** Show the compact form for the tangent sum and difference formulas.
The formulas just developed are frequently called the **Addition Formulas.** Next the so called **Double Angle Formulas** are developed.

\[
\sin 2x = \sin (x + x) = \sin x \cos x + \cos x \sin x
\]
and thus

\[
\sin 2x = 2 \sin x \cos x
\]

The sine of twice an angle equals two times the sine of the angle times the cosine of the angle.

\[
\cos 2x = \cos (x + x) = \cos x \cos x - \sin x \sin x
\]
and thus

\[
\cos 2x = \cos^2 x - \sin^2 x
\]

The cosine of twice an angle equals the cosine squared of the angle minus the sine squared of the angle.

Actually there are two other forms for this identity. Using the Pythagorean Identity, \( \sin^2 x + \cos^2 x = 1 \) and the identity above, complete the exercise below.

**Exercise:** Show that \( \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \) and thus

\[
\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x
\]

Finally \( \tan 2x = \tan (x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} \)
and thus

\[
\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}
\]

The tangent of twice an angle equals twice the tangent of the angle divided by the quantity one minus the tangent squared of the angle.

Finally we derive a set of identities called the **Half-Angle Formulas.**
Since \( \cos 2x = 1 - 2 \sin^2 x \), then \( 2 \sin^2 x = 1 - \cos 2x \) and thus \( \sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}} \).

Replacing \( x \) by \( \frac{x}{2} \) yields

\[
\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}
\]

**The sine of half an angle equals plus or minus the square root of the fraction one minus the cosine of the angle, over two.**

**Note:** The sign of the value depends on the quadrant in which \( \frac{x}{2} \) terminates. Recall that All functions are positive in QI, \( \sin \) & \( \csc \) in QII, \( \tan \) and \( \cot \) in QIII and \( \cos \) and \( \sec \) in QIV. The phrase, "All Students Take Calculus," is a memory aid.

**Exercise:** Use the identity, \( \cos 2x = 2 \cos^2 x - 1 \), to develop the cosine half-angle formula, \( \cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \) and then write what it says in words.

The Tangent Half-Angle Formula can be found by writing

\[
\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\pm \sqrt{\frac{1 - \cos 2x}{2}}}{\pm \sqrt{\frac{1 + \cos 2x}{2}}} = \pm \frac{1 - \cos x}{\sqrt{1 + \cos x}}
\]

but this is not the form generally used. Complete the exercise below to show that

\[
\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}
\]

**Exercise:** Take the fraction \( \pm \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}} \) and first rationalize the numerator and then rationalize the denominator to derive the identity seen above. **Note:** You will need to resolve the ambiguity in sign.
There are other identities that you will verify in the problem set for this section. Identities can be used to find exact trig values for angles other than the special 30, 45 and 60 degree angles we have worked with before. For example, suppose you want the value for \( \sin 15^\circ \). One method to determine this value makes use of addition formulas and another method uses half-angle formulas. This work is shown below.

\[
\sin 15^\circ = \sin (45^\circ - 30^\circ) = \sin 45\cos 30 - \cos 45\sin 30 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

\[
\sin 15^\circ = \sqrt{\frac{1 - \cos 30}{2}} = \sqrt{\frac{1 - \sqrt{3}/2}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2}
\]

Now, except for practicing working with identities, this application is not particularly useful. On the other hand, identities are very useful in calculus when integrating trig expressions. They are also useful in solving equations.

**TRIG EQUATIONS**

You have already solved simple trig equations such as \( \sin x = \frac{1}{2} \). The solution is \( x = \frac{\pi}{6} + 2k\pi \) or \( \frac{5\pi}{6} + 2k\pi \). Notice there are an infinite number of solutions to this equation. Now consider the equation \( \sin x = \cos x \). One way to solve this equation is to divide through by \( \cos x \), producing \( \tan x = 1 \) and hence \( x = \frac{\pi}{4} + 2k\pi \) or \( \frac{5\pi}{4} + 2k\pi \). This procedure made use of the identity \( \tan x = \frac{\sin x}{\cos x} \). Here is another equation. \( \sin 2x = \cos x \)

Using the double angle sine identity yields \( 2 \sin x \cos x = \cos x \)

Combine terms \( 2 \sin x \cos x - \cos x = 0 \)

Factor \( \cos x \cdot (2 \sin x - 1) = 0 \)

Set each factor equal to 0 \( \cos x = 0 \) or \( 2 \sin x - 1 = 0 \)

Thus \( \cos x = 0 \) or \( \sin x = \frac{1}{2} \)

And the solutions are \( x = 2k\pi, \frac{\pi}{6} + 2k\pi \) or \( \frac{5\pi}{6} + 2k\pi \)

You notice that in addition to using your knowledge of trig values and identities, you also utilize algebraic equation solving methods. You will have an opportunity to solve more of these type of trig equations in the problem set that follows.
Problem Set 8.3

1. One definition of a nautical mile is that it is the length of one minute \( \frac{1}{60}^o \) of the circumference of the Earth. Using 3960 miles for the radius of the Earth, calculate the number of feet in a nautical mile.

   **Note:** Since the Earth is not a perfect sphere, its circumference and consequently the length of a nautical mile, depends on where you measure it. Today we agree on an official length for a nautical mile as seen below in problem 3.

2. If you are standing at the equator of the Earth, at what speed (m.p.h) are you moving due to the fact that the Earth revolves on its axis once every 24 hours.

3. If you are in the Space Navy on a ship in orbit about the Earth at an altitude of 100 nautical miles and each orbit takes 2 hours, what is your speed in **knots**?

   **Note:** The term **knot** came into use when sailing ships traversed the oceans. Sailors would **log the speed** using a speed-measuring device consisting of a piece of wood tied on a line which was wound up on a reel. The wood chip often called a log would be dropped into the water and the drag created by the ship moving forward would pull out the line. Knots were tied in the line at intervals of 47 feet 3 inches. The line was allowed to run out for 28 seconds. Officially today, a nautical mile is 1852 meters or about 6,076.10333 feet. The ratio of twenty-eight seconds to one hour is equal to the ratio of 47 feet 3 inches to 6,076.1033 feet. Therefore, every knot the log pulled out in the 28 second time interval represented one nautical miles per hour of speed.

4. A tire company measures tire wear by running a set of 24 inch diameter tires on a car at an average of 2000 rpm for 48 hours. How far did the tires go? At what average speed did the car travel?

5. Use the process used in this section to derive the sine addition formula, to derive the cosine addition formula.

6. In some aeronautical applications of calculus, it is necessary to convert a product of trig functions to a sum. Use the addition formulas to derive the following product to sums formulas.

   \[
   \text{(a) } \sin x \sin y = \frac{1}{2} \left[ \cos (x-y) - \cos (x+y) \right]
   \]
(b) \[ \sin x \cos y = \frac{1}{2} [\sin (x+y) + \sin (x-y)] \]

(c) \[ \cos x \cos y = \frac{1}{2} [\cos (x+y) + \cos (x-y)] \]

7. In other applications of calculus, it is necessary to "break down" even powers of trig functions to first powers only. Show that each of the following is an identity:

(a) \[ \sin^2 x = \frac{1}{2} (1 - \cos 2x) \]  
(b) \[ \cos^2 x = \frac{1}{2} (1 + \cos 2x) \]  

8. Derive an equivalent "1st powers only" expression for (a) \( \sin^4 x \) (b) \( \cos^4 x \).

9. Other applications require that trig functions of multiples of \( x \) be converted to expressions that involve only functions of \( x \). Convert each of the following to an equivalent expression involving powers of \( \sin x \) and/or \( \cos x \). (a) \( \sin 3x \) (b) \( \cos 4x \)

10. Solve each of the following equations on \([0,2\pi)\).

(a) \[ 2 \cos x + \sqrt{3} = 0 \]  
(b) \[ 2 \sin^2 x + \sin x = 0 \]  
(c) \[ 2 \sec^2 x - 5 \sec x + 2 = 0 \]

(d) \[ \tan^2 x - \sec x - 1 = 0 \]  
(e) \[ \cos 4x - \sin 2x = 0 \]  
(f) \[ \tan \frac{1}{2} x + 1 = \cos x \]
8.4 PARAMETRIC AND POLAR EQUATIONS

Previously we have been examining what are called **Plane Cartesian or Rectangular Equations**, so named because they generate rectangular coordinate ordered pairs which can be graphed in the Cartesian Plane. In this section we will develop two other types of equations, each of which generates points that can be graphed in the Cartesian Plane, but neither of which generate rectangular coordinate ordered pairs, and hence are not Cartesian or Rectangular Equations.

PARAMETRIC EQUATIONS

Imagine a projectile fired from a cannon to have the path shown in the picture below.

The projectile's horizontal distance from its starting point (the origin in the picture) is called its **range** and the projectile's vertical distance above its starting point is called its **altitude**. As represented in the picture, when the projectile is at P, its range is x and its altitude is y and we say the ordered pair (x,y) represents the location of the projectile. Now the location of the projectile depends on time, and so we say \( (x,y) = f(t) \). As we saw when introducing trig functions, this is an inconvenient situation having one independent variable but two dependent variables. Recall that we resolved our trig problem by creating two functions such that \( x = \cos \theta \) and \( y = \sin \theta \). We can now do the same thing for our projectile problem by writing \( x = f_1(t) \) and \( y = f_2(t) \). Sets of equations such as these are called **parametric equations**, because two or more variables are each functions of some common variable, called the **parameter**. For our projectile problem the parameter is time (represented by \( t \)).

As you probably realize from thinking about the projectile problem, parametric equations are utilized in Ballistic Problems. More generally, parametric equations are often used to **paramatize a curve**. Consider the graph of the circle \( x^2 + y^2 = 9 \) shown below.
Recall that we can measure an arc of a circle in the same way that we measure the central angle that intercepts the arc. Thus we can write the parametric equation for our circle as

\[ x = 3 \cos t \]
\[ y = 3 \sin t \]

These parametric equations will generate ordered triples \((t, x, y)\). Using the second and third coordinates to produce ordered pairs will, when the ordered pairs are plotted, produce the circle graph shown above. Many parameterizations make use of trig functions, such as you saw in the example above and will see again in the problem set for this section.

One final thought is appropriate at this time. You should realize that the Cartesian Equation that generates a circle, is not a function and, like other non-functions in standard form, can not be graphed on a graphing calculator. Parametric equations can easily be graphed on a graphing calculator and thus parameterization provides an easy method with which to graph non-functions.

POLAR EQUATIONS

Polar Equations are another type of non-Cartesian equations. The descriptor, polar, stems from the concept of Polar Coordinates. The Cartesian coordinates of a point are similar to street or city directions, i.e. they indicate horizontal and vertical directions as to how to move from the origin to the specified point. Polar coordinates, however, are similar to country or field directions, i.e. they indicate shortest distance or straight directions as to how to move from the origin to the specified point. Typically, polar coordinates are ordered pairs, \((r, \theta)\) which indicate that some specified point is some directed distance and some directed angle from the origin which is called the Pole (hence the name of this class of equations defined below).

Polar Equations are equations which generate polar coordinate ordered pairs, \((r, \theta)\).

Now polar equations do produce points that can be graphed in the Cartesian Plane, but the polar coordinates of a point in the plane are not the same numbers as the corresponding Cartesian coordinates of the point. And, in fact, unlike with Cartesian coordinates, there are an infinite number of polar coordinates for any point in the plane. However, we can easily convert between polar and Cartesian coordinates. First, examine the picture shown below.

\[
\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta \\
\cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta \\
x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2} \\
\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \frac{y}{x}
\]

The conversion formulas seen above can also be used to convert Cartesian to polar and polar to Cartesian equations as shown below.
In the first example above, notice that $r = 3$ or $-3$. First of all, since $\theta$ is not in the equation, then the only requirement for a point to be on the graph is that it be ±3 units from the origin. Now with regard to the ±, in polar coordinates, $r$ as well as $\theta$ can be negative. This concept can be grasped by considering polar coordinates $(r, \theta)$ to be directions which say:

*Face in the direction $\theta$ and then go $r$ units forward if $r > 0$ and backward if $r < 0$.*

In the second example above, notice the Cartesian equation is that for a circle with center on the $y$-axis (confirm this!).

In the third example above, the polar equation is more complicated than is the Cartesian equation, but in the first example the Cartesian equation is more complicated than is the polar equation. These observations suggest why it is useful to have different types of equations (including parametric equations).

This section concludes with some pictures of graphs of some common polar equations, along with some generalizations.

**CIRCLES**

- $r = k$
  - center Pole $(0,0)$
- $r = 2a \cos \theta$
  - center $(a,0)$
  - $a > 0$, right circle
  - $a < 0$, left circle
- $r = 2a \sin \theta$
  - center $(0,a)$
  - $a > 0$, top circle
  - $a < 0$, bottom circle

**NOTE:** Polar equations of circles with centers not on the axes, and/or which do not pass through the pole can be complicated.
**CARDIOIDS**

\[ r = 1 + \sin \theta \]
- Cusp points up
- General form: \( r = a - a \sin \theta \)

\[ r = 1 - \sin \theta \]
- Cusp points down
- General form: \( r = a + a \sin \theta \)

\[ r = 1 + \cos \theta \]
- Cusp points right
- General form: \( r = a + a \cos \theta \)

\[ r = 1 - \cos \theta \]
- Cusp points left
- General form: \( r = a - a \cos \theta \)

**Note:** These shapes are called Cardioids because they are heart-shaped.

**SINE ROSES**

\[ r = \sin 2\theta \]
- 4 leaves
- Leaf diameter = 1
- 1st leaf at \( \frac{\pi}{4} \)
- 90° angle between leaves

\[ r = \sin 3\theta \]
- 3 leaves
- Leaf diameter = 1
- 1st leaf at \( \frac{\pi}{6} \)
- 120° angle between leaves

\[ r = \sin 4\theta \]
- 8 leaves
- Leaf diameter = 1
- 1st leaf at \( \frac{\pi}{8} \)
- 45° angle between leaves

**Exercise A:** Make a chart similar to that for the Sine Roses but for the Cosine Roses:
\[ r = 2 \cos 3\theta \], \( r = 2 \cos 4\theta \), \( r = 2 \cos 5\theta \)

**COSINE ROSES**

**Exercise B:** Write some general conjectures concerning roses.
There are many other pretty and interesting polar graphs, but these three sets will serve as an introduction to polar graphing. You will work with these classes of polar graphs as well as some others such as the **Spiral of Archimedes** in the problem set which follows.

**Problem Set 8.4**

1. Consider the parametric equations \( \begin{cases} x = 18t^2 - 2 \\ y = 9t \end{cases} \) and fill in the table shown below.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>-1</td>
<td>2/3</td>
<td>-1/3</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>y</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Plot the (x,y) points generated and then guess the nature of the graph for \(-\infty \leq t \leq \infty\).

(b) Eliminate the parameter t and then find the Cartesian equation.

(c) Graph this equation and see if your guess in part a was correct.

2. Show that one of the parametric representations for \( \frac{x^2}{9} + \frac{y^2}{25} = 1 \) is given by

\( \begin{cases} x = 3 \cos t \\ y = 5 \sin t \end{cases} \). Draw the graph showing ordered triples (t, x, y) for the intercept points of the graph.
3. Sketch the graph of \( \begin{align*} x &= 8 \cos^3 t \\
y &= 8 \sin^3 t \end{align*} \) showing ordered triples \((t, x, y)\) for the intercept points of the graph. Use the Pythagorean Theorem to eliminate the parameter and find the Cartesian equation.

4. Convert each of the following Cartesian equations to polar equations and then sketch the graph.
   
   (a) \( x^2 + y^2 = 25 \)  
   (b) \( x^2 + y^2 - 6y = 0 \)  
   (c) \( 2x + 3y - 6 = 0 \)

5. Graph each of the following polar equations.
   
   (a) \( r = 2 - 2 \cos \theta \)  
   (b) \( r = 3 \sin 5 \theta \)  
   (c) \( r = 2 \sec \theta \)  
   (d) \( r = -6 \sin \theta \)  
   (e) \( r = 2 - 4 \sin \theta \)  
   (f) \( \theta = 2 \)  
   (g) \( r = \csc (\theta + \frac{\pi}{6}) \)  
   (h) \( r = \theta \) [Spiral of Archemedes]

6. Consider what you did in problems 4c and 5g. In general, the polar equation of a line is \( r = \frac{c}{b \sin \theta + a \cos \theta} \) because for this equation, \( ar \cos \theta + br \sin \theta = c \) and thus \( ax + by = c \) which is the equation of a line. Show that the Cartesian equation equivalent to the polar equation of problem 5g is \( x + \sqrt{3} \ y - 2 = 0 \). In general the polar equation of a line \( r = \frac{c}{b \sin \theta + a \cos \theta} \) can be written as \( r = k \csc(\theta + \alpha) \) where \( k = \frac{c}{\sqrt{a^2+b^2}} \). Use the figure shown below to figure out why this is so.

\[
\begin{array}{c}
\sqrt{a^2+b^2} \\
\alpha \\
a \\
b
\end{array}
\]
8.5 VECTORS AND COMPLEX NUMBERS

Up until this point we have been working with quantities such as time, length, mass etc., which have only one measure called **magnitude**. Although each of these quantities can be measured using different units, we talk about time intervals, distance, etc. as being a number assigned to the quantity as its measure. Numerical quantities such as those mentioned above are called **scalars**. Now some quantities, such as displacement, velocity, force, etc., have both magnitude and direction and are classified as **vectors**.

Geometrically, a vector is represented as a directed line segment in the plane as shown below.

Point P is the starting point and Point Q is called the terminal point. The vector can be named using its endpoints as $\vec{PQ}$ or by using a single letter, say $\vec{v}$, with an arrow over it as $\vec{v}$ or in the context of vectors as just as a letter, say $\vec{v}$. The magnitude of $\vec{v}$ is denoted by $|\vec{v}|$ and equals $\sqrt{x^2 + y^2}$. The direction of $\vec{v}$ equals $\tan^{-1}\frac{y}{x}$. Another specialized way that a vector can be represented is called **component form**. The horizontal movement from P to Q in the vector pictured above is $x = c - a$ and the vertical movement is $y = d - b$. Thus the component form of $\vec{v}$ is $(x, y) = (c - a, d - b)$ where $x$ is referred to as $\vec{v}$'s horizontal component and $y$ as $\vec{v}$'s vertical component. Any vectors which have the same components are **equal vectors**.

Algebraically, vectors are added and subtracted by adding and subtracting their components. For example, if $\vec{u} = (-1, 2)$ and $\vec{v} = (5, 1)$ then

$$\vec{u} + \vec{v} = (-1 + 5, 2 + 1) = (4, 3) \text{ and } \vec{u} - \vec{v} = (-1 - 5, 2 - 1) = (-6, 1).$$

Geometrically, vectors are added and subtracted using the **head to toe** method. In the picture shown below, $\vec{u} + \vec{v}$ with $\vec{u}$ starting at the origin is represented. Notice that the "toe" of $\vec{v}$ is at the "head" of $\vec{u}$.
Similarly, \( \mathbf{u} - \mathbf{v} \) is represented in the picture below. Observe that, as with arithmetic, \( \mathbf{u} - \mathbf{v} = \mathbf{u} + \mathbf{-v} \).

\[
\begin{array}{c}
(0,0) \\
(-6,1) \quad \text{\textbullet} \\
(-1,2) \quad \text{\textbullet} \\
(0,0) \\
\end{array}
\]

Now most vector quantities are denoted by their magnitude and direction rather than in component form. For example, suppose a plane is flying at 500 mph on a course of 030° T (30° East of due North - also denoted by N 30° E). This vector is shown below.

\[
\begin{array}{c}
30° \\
500 \\
\text{x} \\
\text{y} \\
\end{array}
\]

Observe that the angle in the triangle is 60° and thus the component form for the vector is \((500 \cos 60°, 500 \sin 60°) = (250, 250\sqrt{3})\). Now suppose a wind is blowing on the plane referenced above, at 40 mph in the direction 060° T. These two vectors are graphed head to toe in the diagram shown below.

\[
\begin{array}{c}
\theta_r \\
500 \\
30° \\
\text{x} \\
\end{array}
\]

From the diagram, you can see that the wind would push the plane along a little faster, but to the right. Vector \( \mathbf{r} \) would represent the resulting speed and direction of the plane,
so it is called the **resultant vector**. For many types of vector problems, you are asked to find a resultant vector.

Observe that the vectors for the plane and wind along with the resultant vector form a triangle. Consequently the trig process of solving triangles can be employed. Shown below is the vector triangle seen above.

![Vector Triangle](image)

Observe that angle B is composed of an angle equal to 30° (by alternate interior angles formed by parallel lines) plus an angle supplementary to 60° and hence equals 150°.

So we have a triangle for which we know SAS. Thus using the Law of Cosines,

\[
r^2 = 500^2 + 40^2 - 2 \cdot 500 \cdot 40 \cdot \cos 150° \approx 286241
\]

and \( r \approx 535.01 \)

To find the direction for the resultant, first find angle A using the Law of Sines.

\[
\frac{\sin A}{40} = \frac{\sin 150°}{535.01} \Rightarrow \sin A = 0.03738318 \Rightarrow A = 2.14°
\]

The resultant angle \( \theta_r \), is then \( = 90° - 30° - 2.14° = 57.86° \)

The process seen above, is used to find the resultant for the sum or difference of two vectors. For a problem in which the resultant for more than two vectors is desired, you could find the resultant for two of the vectors and then use that resultant with another of the vectors and continue this process. However, generally when a resultant for more than two vectors is desired, it is easier to use the three step process listed below.

First, find the component form for each vector
Second, sum components to find the components for the resultant
Third, determine the magnitude and direction for the resultant using its components.

You will see some typical vector problems in the problem set for this section.
COMPLEX NUMBERS

You were introduced to complex numbers earlier in this book. Recall that a complex number is of the form \( \mathbf{a} + \mathbf{b}i \) where \( \mathbf{a} \) and \( \mathbf{b} \) are real numbers and \( i = \sqrt{-1} \). Observe that if \( \mathbf{b} = 0 \), the complex number is a real number and could be graphed on the real number line. On the other hand, non-real complex numbers can not be graphed on the real number line. Complex numbers are graphed in the Complex Plane, which is formed by intersecting a horizontal real number line with a vertical pure imaginary line. A complex number \( \mathbf{a} + \mathbf{b}i \), is graphed as an ordered pair \( \mathbf{a}, \mathbf{b} \), where, as with graphing in the real plane, the first coordinate indicates the horizontal movement from the vertical axis and the second coordinate indicates the vertical movement from the horizontal axis. For example, \( 3 - 4i \) is graphed below.

\[
\begin{array}{c}
y \\
\downarrow \\
(3, -4) \\
\end{array}
\]

From the discussion above, in the context of complex numbers, it is reasonable to say that \( \mathbf{(a, b)} \) represents \( \mathbf{a} + \mathbf{b}i \). Now as you have seen before, addition of two complex numbers is performed as if they were normal algebraic binomials.

For example, \( \mathbf{(a + b i)} + \mathbf{(c + di)} = \mathbf{(a + c)} + \mathbf{(b + d)i} \).
Thus \( \mathbf{(a, b)} + \mathbf{(c, d)} = \mathbf{(a + c, b + d)} \)

The work with complex numbers in coordinate form looks just like our work with vectors in component form, so we now have the understanding that, depending on the context, we can use an ordered pair \( \mathbf{a}, \mathbf{b} \) to represent coordinates of a point in the Cartesian plane, the component form for a vector, the component form for a complex number, or the coordinates of a point in the complex plane.

Now in the last section we saw how to convert Cartesian coordinates to polar coordinates. It stands to reason that we can do the same thing with the coordinate form for a complex number and thus find the polar form of a complex number as shown below.

\[
\begin{array}{c}
y \\
\downarrow \\
\mathbf{P} (x, y) \\
\end{array}
\]

\( \mathbf{P} \) is the graph of \( x + yi \)

\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta , \text{ so } x + yi &= r(\cos \theta + i \sin \theta).
\end{align*}
Note: As with vectors, the length of \( r \) is called the magnitude of the complex number and is indicated by enclosing the complex number in absolute values as \( |x + yi| \).

Example: The process of converting \( \sqrt{3} + i \) to polar form is shown below.

\[
\begin{align*}
\sqrt{3} + i &= 2(\cos 30 + i \sin 30) \\
\end{align*}
\]

Earlier in the book, we discussed the arithmetic of the standard \( a + bi \) form of complex numbers. Recall that the arithmetical operations of addition, subtraction and multiplication are performed as if complex numbers were normal algebraic binomials and that division is performed by multiplying by the conjugate of the denominator. Now, polar form could be used for doing addition and subtraction of complex numbers, but using standard \( (a + bi) \) form is easier. It is also easier to do multiplication or division with complex numbers in standard form, rather than to convert them to polar form, multiply or divide them, and then convert back to standard form. However, the multiplication process in polar form leads to a procedure for finding powers and roots of complex numbers.

Consider the product of two complex numbers expressed in polar form.

\[
r(\cos \alpha + i \sin \alpha) \cdot s(\cos \beta + i \sin \beta).
\]

which equals \( rs(\cos \alpha \cos \beta + i \cos \alpha \sin \beta + i \sin \alpha \cos \beta + i^2 \sin \alpha \sin \beta) \)

which equals \( rs[ (\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \)

which equals \( rs[\cos (\alpha + \beta) + i \sin (\alpha + \beta)] \)

Observe that:

**To multiply, using polar form, you simply multiply the magnitudes and add the angles!**

Exercise A: Show \( \frac{r(\cos \alpha + i \sin \alpha)}{s(\cos \beta + i \sin \beta)} = \frac{r}{s} [\cos(\alpha-\beta) + isin(\alpha-\beta)] \)

To divide, divide the magnitudes and subtract the angles!
Exercise B: Use the product rule, developed above, repeatedly to convince yourself that:

\[ [r(\cos \alpha + i \sin \alpha)]^n = r^n (\cos n\alpha + i \sin n\alpha) \]

The result you just verified is called DeMoivre's Theorem after the English-raised Frenchman Abraham DeMoivre who is credited with discovering the theorem.

Now to develop the formula for finding roots, suppose that

\[ r(\cos \alpha + i \sin \alpha) \text{ is an nth root of } s(\cos \beta + i \sin \beta). \]

Then

\[ [r(\cos \alpha + i \sin \alpha)]^n = s(\cos \beta + i \sin \beta) \]

and

\[ r^n (\cos n\alpha + i \sin n\alpha) = s(\cos \beta + i \sin \beta) \]

Now the only way this can be true is for

\[ r = s^{1/n} = n \sqrt[n]{s} \]

and

\[ n\alpha = \beta + 2k\pi \]

and thus

\[ \alpha = \frac{\beta}{n} + \frac{2k}{n} \pi \quad k = 0,1,\cdots, n-1 \]

An example will illustrate the process and show an application.

Example: Find all the zeros for the equation \( x^4 + 1 = 0 \).

\[
x = \sqrt[4]{-1} = 1 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)
\]

\[
(-1)^{1/4} = 1^{1/4} \left( \cos \left[ \frac{\pi}{4} + k \frac{\pi}{2} \right] + i \sin \left[ \frac{\pi}{4} + k \frac{\pi}{2} \right] \right)
\]

\[
= 1(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}), 1(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}), 1(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}), 1(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})
\]

\[
= \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}, \text{ and } \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}.
\]

Observe that there are four zeros as repeated application of the Fundamental Theorem of Algebra tell us there should be. You will have a few problems dealing with complex numbers in the problem set for this section.

Problem Set 8.5
1. In the plane and wind example seen in this section, the two vectors and the resultant vector formed an oblique triangle. Solve this triangle and find the actual speed and direction the plane takes.

2. If the plane and wind you worked with in Problem 1 are placed starting at the origin, then the resultant vector is the diagonal of the parallelogram formed by two sets of plane and wind vectors as shown below.

```
(p_x, p_y) (R_x, R_y)
```

Use the information given in the example, to show that

\[ R_x = P_x + W_x \quad \text{and} \quad R_y = P_y + W_y. \]

3. The following set of forces act on an object.

\[ F_1 = 30 \text{ lbs at } 60^\circ, \quad F_2 = 40 \text{ lbs at } 150^\circ, \quad F_3 = 50 \text{ lbs at } 225^\circ, \quad F_4 = 60 \text{ lbs at } 330^\circ. \]

Find the magnitude and direction of the resulting force.

4. A plane is scheduled to make a 2 hour flight from City A to City B which is 1000 miles Northeast of A. An Easterly wind is blowing at a constant rate of 50 mph. What course and speed should the pilot set in order to make straight for B and make the trip in the scheduled time?

5. Divide \(-6\) by \(1 + \sqrt{3}i\) in rectangular form. Then rework this problem by converting to polar form, dividing, and then converting back to rectangular form. Did you get the same result?

6. Evaluate \((\sqrt{3} + i)^4\) in rectangular form. Then rework this problem by converting to polar form, using DeMoivre's Theorem, and then converting back. Did you get the same result?

7. Find the indicated roots: (a) \((-8 - 8\sqrt{3}i)^{1/4}\) (b) \((-27)^{1/3}\)

8. Find all the zeros, in polar form, for \(x^5 - 243 = 0\).