General Directions: Show all work analytically. When determining the convergence or divergence of a series, state the test (one on the handout) you use and analytically show in detail how you use it. If asked to approximate the sum and/or discuss a bound on the error for an alternating series, assume the alternating series satisfies the conditions of the Alternating Series Test. Summarize your conclusions in verbal sentence form.

1. Analytically determine if the sequence \( \left\{ n \ln\left(1 + \frac{3}{n}\right) \right\} \) converges or diverges. Identify indeterminate forms and show your work analytically. (9 points)

   Let \( f(x) = x \ln\left(1 + \frac{3}{x}\right) \). Then

   \[
   \lim_{x \to \infty} x \ln\left(1 + \frac{3}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \left( \frac{1}{1 + \frac{3}{x}} \right) \left( \frac{3}{x^2} \right) = \lim_{x \to \infty} \frac{3}{1 + 3/x} = 3
   \]

   The sequence converges to 3.

2. Use the Direct Comparison Test to analytically determine if the series \( \sum_{n=1}^{\infty} \frac{\sin n + 1}{n^2} \) converges or diverges. Show your work analytically. (9 points)

   \[
   \sin n \leq 1 \quad \text{for } n \geq 1
   \]

   \[
   \sin n + 1 \leq 2
   \]

   \[
   \frac{\sin n + 1}{n^2} \leq \frac{2}{n^2}
   \]

   \[
   \sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
   \]

   \[
   \frac{\pi^2}{6} \quad \text{is a convergent } p \text{-series } (p=2>1)
   \]

   By the Direct Comparison Test, the series converges.
3. Determine if the series \( \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n} \) is absolutely convergent, conditionally convergent, or divergent. Show all work analytically. (10 points)

First check for absolute convergence:

\[
\left| \frac{(-1)^n n^3}{4^n} \right| = \frac{n^3}{4^n} \quad \text{since } n \geq 1
\]

Using the Ratio Test,

\[
\lim_{n \to \infty} \frac{(n+1)^3}{4^{n+1}} = \lim_{n \to \infty} \frac{(n+1)^3}{4^n} \cdot \frac{4^{n-1}}{n^3}
\]

\[
= \lim_{n \to \infty} \frac{3}{4} \cdot \frac{n+1}{n}
\]

\[
= \frac{3}{4}
\]

\(0 < \frac{3}{4} < 1\)

By the Ratio Test, the series converges absolutely.
For problems 4-6, use any test other than the integral test to determine if the series converges or diverges. If a series is geometric and convergent, find its sum. State which test you use and show analytical work to justify your result. (9 points each)

4. \[ \sum_{n=5}^{\infty} \frac{2n^5 + 3}{n^5 - 4n^4} \]

\[ \lim_{n \to \infty} \frac{2n^5 + 3}{n^5 - 4n^4} = \lim_{n \to \infty} \frac{2 + \frac{3}{n^5}}{1 - \frac{4}{n}} \]

\[ = 2 \neq 0 \]

By the Divergence Test, the series diverges.

5. \[ \frac{1}{4} - \frac{3}{16} + \frac{9}{64} - \frac{27}{256} + \cdots \] (State the nth term of the series.)

\[ a_n = \frac{3^{n-1}}{4^n}, \quad a = \frac{1}{4}, \quad r = -\frac{3}{4} \]

Since \[ |\frac{-3}{4}| < 1 \], the geometric series converges.

Its sum is \[ S = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-(-\frac{3}{4})} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}. \]
6. \( \sum_{n=0}^\infty \frac{\sqrt{n} + 3}{2n^2 + 4} \)

\[
\frac{T_n}{n^2} = \frac{\frac{n^{1/2}}{n^2}}{\frac{n^{1/2}}{n^2}} = \frac{1}{n^{3/2}}
\]

Compare to the convergent p-series, \( \sum_{n=1}^\infty \frac{1}{n^{3/2}} \), \( p = \frac{3}{2} > 1 \).

\[
\lim_{n \to \infty} \frac{T_n + 3}{2n^2 + 4} = \lim_{n \to \infty} \frac{n^{1/2} + 3}{2n^2 + 4} \cdot \frac{n^{3/2}}{1}
\]

\[
= \lim_{n \to \infty} \frac{n^{2} + 3n^{3/2}}{2n^2 + 4}
\]

\[
= \lim_{n \to \infty} \frac{1 + \frac{3}{n^{1/2}}}{2 + \frac{4}{n^{2}}}
\]

\[
= \frac{1}{2}
\]

\( 0 < \frac{1}{2} < \infty \)

By the limit comparison test, the series converges.

\[
\text{OR} \quad \lim_{n \to \infty} \frac{\frac{1}{n^{3/2}}}{\frac{T_n + 3}{2n^2 + 4}} = \lim_{n \to \infty} \frac{2n^2 + 4}{T_n + 3} \cdot \frac{1}{n^{3/2}}
\]

\[
= \lim_{n \to \infty} \frac{2n^2 + 4}{n^2 + 3n^{3/2}}
\]

\[
= \lim_{n \to \infty} \frac{2 + \frac{4}{n^2}}{1 + \frac{3}{n^{1/2}}}
\]

\[
= 2
\]
7. Use the integral test for the series \( \sum_{n=1}^{\infty} \frac{n}{n^4 + 1} \) to determine if the series converges or diverges.

Show all work analytically. (10 points)

\[
\text{Let } f(x) = \frac{x}{x^4 + 1}
\]

1. \( f(x) \) is positive for \( x \geq 1 \)
2. \( f(x) \) is continuous for \( x \geq 1 \)
3. \( f(x) \) is decreasing for \( x \geq 1 \)

\[
f'(x) = \frac{1-3x^4}{(x^4+1)^2} \quad 1-3x^4 = 0 \quad 3x^4 = 1 \quad x = \pm \frac{1}{\sqrt[3]{3}}
\]

Apply the Integral Test,

\[
\int_{1}^{\infty} \frac{x}{x^4 + 1} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{x^4 + 1} \, dx
\]

\[
= \frac{1}{2} \lim_{b \to \infty} \left( \arctan(x^2) \right)_{1}^{b}
\]

\[
= \frac{1}{2} \lim_{b \to \infty} \left( \arctan(b^2) - \arctan(1) \right)
\]

\[
= \frac{1}{2} \left( \pi/2 - \pi/4 \right)
\]

\[
= \pi/8
\]

Therefore, the improper integral converges.

By the Integral Test, the series converges.
8. Given that the power series \( \sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{2^n(n+1)} \) converges for \( 1 < x < 5 \), determine its interval of convergence. Show your work analytically. Hint: This means check the endpoints. (12 points)

Let \( x = 5 \), \( \sum_{n=0}^{\infty} \frac{(5-3)^{n+1}}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{2}{n+1} \)

Compare this series to the divergent Harmonic Series, \( \sum \frac{1}{n} \)

\[
\lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{n+1}{2n+1} = \frac{1}{2} \quad 0 < \frac{1}{2} < \infty
\]

By the limit Comparison Test, the series diverges at \( x = 5 \).

On \( \lim_{n \to \infty} \frac{\frac{2}{n+1}}{\frac{2}{n}} = \lim_{n \to \infty} \frac{2}{n+1} = 2, \quad 0 < 2 < \infty \)

Let \( x = 1 \), \( \sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2}{n+1} \)

Using the Alternating Series Test, \( b_n = \left| \frac{(-1)^{n+1} 2}{n+1} \right| = \frac{2}{n+1} \)

1. \( \lim_{n \to \infty} \frac{2}{n+1} = 0 \)

2. \( b_{n+1} \leq b_n \) \( \Rightarrow \frac{2}{n+2} \leq \frac{2}{n+1} \)

\( \Rightarrow f(x) = \frac{2}{x+1} \) \( f'(x) = -2 \frac{2}{(x+1)^2} < 0 \) for all \( x \neq -1 \)

By the Alternating Series Test, the series converges at \( x = 1 \).

The interval of convergence for the power series is \( 1 \leq x < 5 \).
9. Analytically find the radius of convergence of the power series \( \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^n \ln n} \). (8 points)

\[
\left| \frac{(-1)^n x^n}{4^n \ln n} \right| = \frac{|x|^n}{4^n \ln n}
\]

Using the Ratio Test,

\[
\lim_{n \to \infty} \left| \frac{\frac{|x|^{n+1}}{4^{n+1} \ln(n+1)}}{\frac{|x|^n}{4^n \ln n}} \right| = \lim_{n \to \infty} \left| \frac{|x|}{\frac{4 \ln n}{\ln(n+1)}} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{|x|}{\frac{4 \ln n}{\ln(n+1)}} \right| = \frac{|x|}{4} \left( \lim_{n \to \infty} \frac{\ln n}{\ln(n+1)} \right) = \frac{|x|}{4}
\]

The power series converges when \( \frac{|x|}{4} < 1 \) or \( |x| < 4 \).

The radius of convergence is \( R = 4 \).

Note: Let \( f(x) = \frac{\ln n}{\ln(n+1)} \)

\[
\lim_{x \to \infty} \frac{\ln(x)}{\ln(x+1)} = \lim_{x \to \infty} \frac{x}{x+1} = \lim_{x \to \infty} \frac{x+1}{x} = 1
\]
Multiple Choice Problems 10-14 (4 points each): Each of the following multiple choice questions has only one solution. Circle the response that best answers the question. If your selection is correct, you will receive full credit (4 pts); if you do not circle any possible responses for a question, you will receive 0 points; and if you select an incorrect response, you will be penalized 1 point, i.e., you will receive a -1 for this question.

10. Determine which of the following series converges.

A) \[ \sum_{n=1}^{\infty} \frac{1}{n} \] Harmonic Series - Diverges

B) \[ \sum_{n=0}^{\infty} 3 \left( \frac{4}{3} \right)^n \] Divergent Geometric Series \[ \left| \frac{4}{3} \right| > 1 \]

C) \[ \sum_{n=0}^{\infty} \frac{(n+1)!}{2^n} \] Diverges (Ratio Test or Divergence Test)

D) \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \] Convergent P-Series, \( p = \frac{3}{2} > 1 \)

E) None of the above.

11. Determine which of the following series diverges.

A) \[ \sum_{n=0}^{\infty} \frac{1}{2^n} \] Convergent geometric Series, \( |\frac{1}{2}| < 1 \)

B) \[ \sum_{n=1}^{\infty} (4+(-1)^n) \] Divergence Test \( \lim_{n \to \infty} (4+(-1)^n) \neq 0 \) oscillates

C) \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \] Converges (Alternating Series Test or Absolute Convergence - Ratio Test)

D) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] Convergent P-Series, \( p = 2 > 1 \)

E) None of the above.
12. Find the partial sum \( s_4 \) of the series \( \sum_{n=1}^{\infty} \frac{1}{2+3^n} \). Give your answer to five decimal places.

A) \( s_4 = 0.32539 \)
B) \( s_4 = 0.33744 \)
C) \( s_4 = 0.03448 \)
D) \( s_4 = 0.34152 \)
E) \( s_4 = 0.01205 \)

\[ S_4 = \frac{1}{2+3} + \frac{1}{2+3^2} + \frac{1}{2+3^3} + \frac{1}{2+3^4} \approx 0.33744 \]

13. What is the fewest number of terms of the series we need to add in order to find the sum of the series to the indicated accuracy?

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \]  

\( |\text{error}| < 0.0399 \)

\[ b_n = \frac{1}{n^2} \]

A) \( n = 5 \)
B) \( n = 12 \)
C) \( n = 13 \)
D) \( n = 6 \)
E) \( n = 8 \)

\[ b_5 = 0.04 \]

\[ b_6 = 0.02778 \approx 0.0399 \]

\[ \frac{10000}{399} < (n+1)^2 \]

\[ \frac{10000}{399} < (n+1)^2 \]

\[ 4.006 < n \Rightarrow n = 5 \]

14. Determine the radius of convergence (R) for the series \( \sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{n!} \).

A) \( R = 1 \)
B) \( R = 0 \)
C) \( R = \infty \)
D) \( R = 2 \)
E) none of the above.

\[ \lim_{n \to \infty} \left| \frac{3^{n+1}(x-2)^{n+1}}{(n+1)!} \right| = \lim_{n \to \infty} \left| \frac{3}{n+1} (x-2) \right| \]

\[ = \frac{3|x-2|}{n+1} \lim_{n \to \infty} \frac{1}{n+1} \]

\[ = 3|x-2| \cdot 0 \]

\[ = 0 \]

The power series converges for all \( x \).