**General Directions:** Show all work analytically. When determining the convergence or divergence of a series, state the test (one on the handout) you use and analytically show in detail how you use it. Summarize your conclusion in verbal sentence form.

For problems 1-5, determine if the series converges or diverges. (9 points each)

1. \[ \sum_{n=1}^{\infty} \frac{3n-1}{5n^2 + 4n + 1} \]

   **Note:** \( \frac{n}{n^2} = \frac{1}{n} \)

   Compare this series to the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \), which diverges.

   \[ \lim_{n \to \infty} \frac{3n-1}{5n^2 + 4n + 1} = \lim_{n \to \infty} \frac{3n^2-n}{5n^2+4n+1} = \frac{3}{5} \text{ where } 0 < \frac{3}{5} < \infty \]

   OR \[ \lim_{n \to \infty} \frac{Y_n}{\frac{3n-1}{5n^2 + 4n + 1}} = \frac{3}{5} \]

   By the Limit Comparison Test, the given series diverges.

2. \[ \sum_{n=1}^{\infty} \frac{n!}{100^n} \]

   **Note:** \( a_{n+1} = \frac{(n+1)!}{100^{n+1}} \)

   Using the Ratio Test,

   \[ \lim_{n \to \infty} \left| \frac{(n+1)!}{100^{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)!}{n!} \cdot \frac{100^n}{100^{n+1}} = \lim_{n \to \infty} \frac{n+1}{100} = \infty > 1 \]

   By the Ratio Test, the given series diverges.
3. \[ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^3} \]

This is a convergent \( p \)-series with \( p = 3 > 1 \).

4. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \] (Use the integral test.)

First check the 3 conditions that must hold to apply the test.

Let \( f(x) = \frac{1}{x^2 + 1} \)

1. \( f(x) \) is continuous for all \( x \) (so specifically for \( x = 1 \))
2. \( f(x) \) is positive for all \( x \) (specifically for \( x = 1 \))
3. \( f(x) \) is decreasing for \( x > 0 \) (specifically for \( x = 1 \))

\[ f'(x) = \frac{-2x}{(x^2 + 1)^2} \quad f'(x) < 0 \text{ for } x > 0 \]

Then

\[ \int_{1}^{\infty} \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2 + 1} \, dx \]

\[ = \lim_{b \to \infty} \left[ \arctan x \right]_{1}^{b} \]

\[ = \lim_{b \to \infty} \left( \arctan b - \arctan 1 \right) \]

\[ = \frac{\pi}{2} - \frac{\pi}{4} \]

\[ = \frac{\pi}{4} \]

By the Integarl Test, the given series converges because the improper integral converges.
5. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{\ln n} \]

Let \( f(x) = \frac{\sqrt{x}}{\ln x} \), then

\[
\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\sqrt{x}}{a} = \infty \neq 0
\]

By the Test for Divergence, the given series diverges.

6. For the series \( \sum_{n=1}^{\infty} 4 \left( \frac{-2}{3} \right)^n \)

a. Write out the first 4 terms. (2 points)

\[
\begin{align*}
A_1 &= -\frac{8}{3} \\
A_2 &= \frac{16}{9} \\
A_3 &= -\frac{32}{27} \\
A_4 &= \frac{64}{81}
\end{align*}
\]

b. Determine if it converges or diverges. If it converges, find its sum. (8 pts)

This is a geometric series where

\[
a = -\frac{8}{3} \text{ and } r = -\frac{2}{3}.
\]

Since \( |r| < 1 \), the geometric series converges.

\[
\frac{a}{1-r} = \frac{-\frac{8}{3}}{1-(-\frac{2}{3})} = \frac{-8}{3} \cdot \frac{3}{5} = -\frac{8}{5}.
\]

The sum of the series is \(-\frac{8}{5}\)
7. Analytically determine whether the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) is absolutely convergent, conditionally convergent, or divergent. Show all work. (12 points)

Check for absolute convergence:

\[
\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \text{This is a divergent } p \text{-series where } p = \frac{1}{2} \leq 1.
\]

Thus the series does not converge absolutely.

Check for conditional convergence by using the Alternating Series Test.

\( b_n = \frac{1}{\sqrt{n}} \)

Then 1. \( \lim_{n \to \infty} b_n = 0 \)

2. Is 0 < \( b_{n+1} \leq b_n \) for all \( n \)? Yes

Let \( f(x) = \frac{1}{x} \) or \( \frac{n+1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n}} \)

\[
f'(x) = \frac{-1}{2} x^{-3/2} < 0 \quad \text{for } x \neq 0.
\]

By the Alternating Series Test, the given series converges. Therefore it is conditionally convergent.
8. For the power series
\[ \sum_{n=1}^{\infty} \frac{(x-2)^n}{n5^n} \]

a. What is the center? (2 points)

The center is at \( a = 2 \).

b. Analytically find the radius of convergence. (10 points)

Using the Ratio Test,
\[
\lim_{n \to \infty} \left| \frac{(X-2)^{n+1}}{(n+1)5^{n+1}} \right| = \lim_{n \to \infty} \frac{|(X-2)^n|}{\frac{n}{5} \cdot \frac{5^n}{n+1}}
\]
\[
= \lim_{n \to \infty} \left| \frac{X-2}{5} \right| \lim_{n \to \infty} \frac{n}{n+1}
\]
\[
= \frac{|X-2|}{5}
\]

By the Ratio Test, the power series converges when \( \frac{|X-2|}{5} < 1 \) or \( |X-2| < 5 \). The radius of convergence is \( r = 5 \).

c. Determine the interval of convergence. Be sure to include a check for convergence at the endpoints of the interval. (6 points)

\( |X-2| < 5 \) \( \Rightarrow \) \( -5 < X - 2 < 5 \) \( \Rightarrow \) \( -3 < X < 7 \)

Check for convergence at the endpoints,

Let \( X = 7 \) \( \Rightarrow \) \( \sum_{n=1}^{\infty} \frac{(7-2)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n} \) (Harmonic Series) Diverges

Let \( X = -3 \) \( \Rightarrow \) \( \sum_{n=1}^{\infty} \frac{(-3-2)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) (Alternating Harmonic Series) Converges

The interval of convergence for the power series is \( -3 \leq X < 7 \).
9. For the function \( f(x) = \cos x \)

a. Find the Maclaurin series expansion through the third non-zero term. Show all work. (6 pts)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{f^{(n)}(0)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>1</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>2</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>3</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>4</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\]

b. Use summation notation and the \( n \)th term to represent the series you found in part a. (4 pts)

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{(Converges for all} \ x) \]

c. Use your series in part b to find a series to represent \( \cos(x^2) \). (4 pts)

\[
\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}
\]

d. Use your answer in part c. to evaluate \( \int_0^1 \cos(x^2) \, dx \) with an error less than 0.001. (6 pts)

\[
\int_0^1 \cos(x^2) \, dx = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n x^{4n}}{(2n)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! (4n+1)}
\]

Using the Alternating Series Estimation Theorem:

\[
\left| \int_0^1 \cos(x^2) \, dx - \sum_{n=0}^{N} \frac{(-1)^n}{(2n)! (4n+1)} \right| \leq \frac{1}{(2(N+1))! (4(N+1))!} < 0.001
\]

for \( N = 3 \), \( b_3 = 0.0011 < 0.001 \). Then

\[
\int_0^1 \cos(x^2) \, dx \approx \sum_{n=0}^{2} \frac{(-1)^n}{(2n)! (4n+1)} \approx 0.90463
\]