1. Give an example and state the nth term for a
   a. Convergent sequence. Verify it converges. (8 points)
      
      \[ \frac{3}{n^3} \]
      
      Verification: \[ \lim_{n \to \infty} \frac{3}{n^3} = 0 \]

   b. Divergent sequence. Verify it diverges. (8 points)
      
      \[ \frac{3}{n^3} \]
      
      Verification: \[ \lim_{n \to \infty} \frac{3}{n^3} = \infty \]

2. For the series \[ \frac{2}{1} + \frac{4}{3} + \frac{8}{7} + \frac{16}{15} + \frac{32}{31} \ldots \]
   a. Determine the general term, \( a_n \), of the series. (5 points)
      
      \[ a_n = \frac{2^n}{2^n - 1} \]

   b. Write out the 3rd partial sum as an exact value. (You do not have to simplify). (5 points)
      
      \[ S_3 = \frac{2}{1} + \frac{4}{3} + \frac{8}{7} = \]

For problems 3-8, state if it is a sequence or series and determine if it converges or diverges.

For a sequence, show it converges by finding its limit. For a series, state the test (one on the handout) you use and show how you use it. Show all work analytically. Summarize your conclusion in sentence form. (8 points each)

3. \[ \sum_{n=1}^{\infty} \frac{n\pi}{5n+1} \]

   \[ \lim_{n \to \infty} \frac{n\pi}{5n+1} = \frac{\pi}{5} \neq 0 \]

   The series diverges by the Test for Divergence.
\[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]

\[ f(x) = \frac{1}{x \ln x} \quad f(x) \text{ is positive, continuous, and non-increasing for } x > e \]

\[ f'(x) = \frac{-(1 + \ln x)}{x^2 (\ln x)^2} < 0 \text{ for } x > e \]

\[ \int_{a}^{b} x \ln x \, dx = \lim_{b \to \infty} \int_{a}^{b} x \ln x \, dx = \lim_{b \to \infty} \left[ \ln (\ln x) \right]_{a}^{b} \]

\[ = \lim_{b \to \infty} \left[ \ln (\ln b) - \ln (\ln 2) \right] = \infty \]

By the Integral Test the series diverges because the improper integral diverges.

5. \[ \left\{ \frac{n!}{(n-2)!} \right\} \]

\[ \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) \]

\[ \lim_{n \to \infty} n(n-1) = \infty \]

The sequence diverges.

6. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}} \]

\[ = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{Divergent } p \text{-series} \]

\[ p = \frac{1}{2} \leq 1 \]

\[ \lim_{n \to \infty} \frac{\sqrt{n}}{n^{1/2}} = \lim_{n \to \infty} \frac{1}{n^{1/2}} = 1 > 0 \]

The series diverges by the Limit Comparison Test.
7. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad \sum \frac{1}{n^{1/2}} \quad p = 1/2 \leq 1 \quad \text{Series diverges.} \]

Note: The series does not converge absolutely. So we must use the Alternating Series Test.

a) \[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \]

b) \[ \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \quad \text{for } n \geq 1 \]

\[ f(x) = x^{-1/2} \]

\[ f'(x) = -\frac{1}{2} x^{-3/2} \leq 0 \quad \forall x \geq 1 \]

So \( f \) is decreasing.

The series converges by the Alternating Series Test, conditionally.

\[ \sum_{n=1}^{\infty} \frac{e^n}{n!} \]

\[ \lim_{n \to \infty} \left| \frac{e^{n+1}}{(n+1)!} \right| = \lim_{n \to \infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} \]

\[ = \lim_{n \to \infty} \frac{e}{n+1} = 0 < 1 \]

The series converges by the ratio test.
9. For the convergent series \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} \), show how many terms are required to compute the sum of the series with an error of less than 0.002. (8 points)

Find the value of \( n \) for which:

\[
|R_n| = |b_{n+1}| = \frac{n+1}{4^{n+1}} \leq 0.002
\]

\( n = 4 \) \( \frac{5}{4^5} \approx 0.005 \leq 0.002 \)

\( n = 5 \) \( \frac{6}{4^6} \approx 0.0014 \leq 0.002 \)

Five terms are required to compute the sum of the series with an error of less than 0.002.

10. Write the number 1.36\overline{36} as an infinite series and express the number as a ratio of integers. Show your work. (8 points)

\[
1.36\overline{36} = 1 + .36 + .0036 + .000036 + \ldots
\]

\[
= 1 + \frac{36}{100} + \frac{36}{10^4} + \frac{36}{10^6} + \ldots
\]

Geometric series with \( a = \frac{36}{100}, r = \frac{1}{10^2}, |r| < 1 \)

\[
\frac{a}{1-r} = \frac{\frac{36}{100}}{1-\frac{1}{10^2}} = \frac{36}{99} = \frac{12}{33}
\]

So \( 1.36\overline{36} = 1 + \frac{12}{33} \)

\[
= \frac{45}{33}.
\]
11. Find the interval of convergence of the power series. Be sure to include a check for convergence at the endpoints of the interval. Show all work analytically. Summarize your conclusion in sentence form. \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n} \) (15 points)

Using the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x-2)^n} \right| = \lim_{n \to \infty} \left| \frac{(x-2) \cdot n}{3(n+1)} \right| = \lim_{n \to \infty} \left| \frac{x-2}{3} \cdot \frac{n}{n+1} \right| = \frac{|x-2|}{3} \lim_{n \to \infty} \frac{n}{n+1} = \frac{|x-2|}{3}
\]

The power series converges when \( \frac{|x-2|}{3} < 1 \)

\( |x-2| < 3 \) \iff \( -3 < x-2 < 3 \) \iff \( -1 < x < 5 \)

The center is 2 and the radius is \( R = 3 \).

Check the endpoints for convergence,

- Let \( x = 5 \) \( \sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \) Harmonic Series Diverges
- Let \( x = -1 \) \( \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) Alternating Harmonic Series Converges

The interval of convergence for the power series is \([ -1, 5 ) \)