CHAPTER 6

ANALYTIC GEOMETRY

1. Linear Equations and Inequalities
2. Functions, Relations and Circle Graphs
3. Coordinate Proofs
4. Quadratic Equations and Inequalities
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6.1 LINEAR EQUATIONS AND INEQUALITIES

INEQUALITIES AND INTERVALS

In your previous work with algebra you learned to solve linear equations in one variable. For example, if \( 2x + 3 = 8 \), then \( x = \frac{5}{2} \). This solution is "graphed" on a number line as a single point. Later we will be examining equations which have more than one solution. Then the solution set would graph as more than one point on the number line.

Closely related to equations are statements called inequalities, which involve the order relations of \(<\) (less than) and \(>\) (greater than) first mentioned in Section 1.3. Consider the inequality \( 2x + 3 > 8 \). As we just saw above, when \( x = \frac{5}{2} \), then \( 2x + 3 = 8 \). Because the coefficient of \( x \) is positive, it is clear that when \( x > \frac{5}{2} \), \( 2x + 3 > 8 \). Thus the solution set for the inequality is \( \{x \mid x > \frac{5}{2} \} \). [ Recall that this set builder notation is read as, \( \text{the set of } x \text{'s such that } x > \frac{5}{2} \).] This solution set graphs as a half-line as shown below.

\[
\begin{array}{c}
\bullet \bullet \bullet -3 -2 -1 0 1 2 3 4 5 6 \bullet \bullet \bullet \\
\end{array}
\]

The graph of this set would contain all points to the right of \( \frac{5}{2} \) on the number line. In Interval Notation, this solution set is indicated by \((\frac{5}{2}, \infty)\), where the "horizontal 8" symbol represents infinity. The infinity symbol is not a number, but rather loosely means goes on forever, in this case to the right.

In a similar manner, the inequality \( 2x + 3 < 8 \) has a solution set \( \{x \mid x < \frac{5}{2} \} \) which graphs as shown below.

\[
\begin{array}{c}
\bullet \bullet \bullet -3 -2 -1 0 1 2 3 4 5 6 \bullet \bullet \bullet \\
\end{array}
\]

The graph of this set would contain all points to the left of \( \frac{5}{2} \) on the number line. In interval notation this solution set is indicated as \((-\infty, \frac{5}{2})\), where negative infinity represents going on forever, in this case to the left.

Now a combination of the relations \(<\) or \(>\) along with the equivalence relation \(=\) is indicated by \( \leq \) or \( \geq \) respectively. For example, a typical inequality is \( 2x + 3 \geq 8 \). The solution set for this inequality is \( \{x \mid x \geq \frac{5}{2} \} \) or in interval notation \([\frac{5}{2}, \infty)\). Likewise
the solution set for \(2x + 3 \leq 8\) is \(\{x \mid x \leq \frac{5}{2}\}\) or in interval notation as \((-\infty, \frac{5}{2}]\). Notice that in both cases, with interval notation, the fact that \(\frac{5}{2}\) is included in the solution set is indicated with a bracket instead of a parentheses. The graphs for these two solution sets would look like the previous graphs, except that the open circles at \(\frac{5}{2}\) would be filled in to indicate that \(\frac{5}{2}\) is included in the solution set. Recall from geometry that these graphs are called rays.

Interval notation is used extensively in mathematics to describe sets of points. Intervals are classified as open, closed or half-open (or half-closed) depending on whether or not endpoints of the interval are included in the set.

**Examples:**
- Open interval \((-2,3)\) represents \(\{x \mid -2 < x < 3\}\).
- Half-Open interval \([-2,3)\) represents \(\{x \mid -2 \leq x < 3\}\).
- Half-Open interval \((-2,3]\) represents \(\{x \mid -2 < x \leq 3\}\).
- Closed interval \([-2,3]\) represents \(\{x \mid -2 \leq x \leq 3\}\).

Now as you probably have suspected, inequalities appear to be solved in a manner similar to that used for equations. For example, \(3x - 5 \geq x + 4 \Rightarrow 2x \geq 9 \Rightarrow x \geq \frac{9}{2}\). There is, however, one important difference between solving inequalities and solving equations.

Consider the inequality \(3 - 4x > x + 13\)
from which \(-5x > 10\).
Now if we divide by \(-5\), \(x > -2\) but this is wrong!
Go back to \(-5x > 10\)
rewrite as \(-10 > 5x\)
and thus \(-2 > x\)
which is equivalent to \(x < -2\) which is correct.

Thus what we should do is:

**Reverse the inequality symbol whenever we divide (or multiply) by a negative number.**
The Street Map above shows the location of Maria's house (M), Bob's house (B), Anthony's house (A), and their school (S).

**Exercise:** Write a set of directions showing how each of them get home from school. For example, to go to school, Anthony would go one block W and then three blocks S (or three blocks S and then one block W).

Previously, you have studied first degree equations and inequalities in one variable, and seen that the solution sets can be graphed on the real number line. In this subsection, you will be extending this development to first degree equations in two variables. A first degree equation in two variables is frequently referred to as a linear equation because its graph is a line. Such a graph is drawn on what is called the Cartesian Plane (named after Rene' Descartes (1596-1650) the French mathematician who devised it). This plane, which is also called the rectangular coordinate plane, is formed by two real number lines intersecting at right angles.

The point of intersection of these two lines is called the origin. The lines themselves are referred to as the coordinate axes or simply the axes. The location of any point in the plane can be described by an ordered pair of real numbers of the form (x, y) such that the first coordinate (called the abscissa) represents the horizontal distance from the vertical...
axis (positive if right, and negative if left), and the second coordinate (called the *ordinate*) the vertical distance (positive if above, and negative if below) from the horizontal axis.

Consider the equation $2x + 3y = 6$.

If $x = 0$, then $y = 2$. If $x = 3$, then $y = 0$. If $x = -3$, then $y = 4$.

Generating more ordered pairs that satisfy the equation, and graphing these "points" produces the table and picture below.
Perhaps you noticed that "x values" were chosen first, and then the corresponding "y values" were determined. Although typical, this choice is arbitrary. However, once it is decided to choose x first, then the horizontal axis is labeled x. The second element of the ordered pair (y in this example) is used to name the vertical axis. The points plotted are then ordered pairs of the form (x,y). The first coordinate, x in this case, is called the abscissa and the second coordinate, y in this case, is called the ordinate. If even more ordered pairs were generated and more points plotted, the points would continue to "line up" and fill in until a line, as shown below, is formed.

This produces a simple procedure for graphing a linear equation: Generate a set of ordered pairs, plot the points, and with a straightedge "lined up" on these points draw the line. This geometric idea is frequently expressed as "two points determine a line." Examples of several equations and their graphs are shown below.

\[
\begin{align*}
\text{x - 3y} &= 6 \\
\text{2x + 5y} &= -10 \\
y &= 5 \\
x &= -4
\end{align*}
\]

Note: It is customary to show coordinates of points on lines where they cross the coordinate axes. Such points are called intercepts. Also, we will omit the arrows and just have the understanding that the lines continue forever.

Questions:

What is the y-coordinate of the x-intercept of a line?

What is the x-coordinate of the y-intercept of a line?

What kind of line would not have an x-intercept?

What kind of line would not have a y-intercept?
Now consider the inequality \( y \geq 2x + 3 \). The graph of \( y = 2x + 3 \) is shown on the left below.

All points on this line have the y-coordinate equal to 3 more than twice the x-coordinate. Some experimenting* reveals that points above this line have y-coordinates greater than 3 more than twice the x-coordinate. Consequently, the graph of \( y \geq 2x + 3 \) (shown in the center figure above) consists of all points in the plane on, and above the line with equation \( y = 2x + 3 \). Similar reasoning shows the graph of \( y < 2x + 3 \) (shown in the right figure above) to be the portion of the plane below the line with equation \( y = 2x + 3 \). Notice that in this graph the boundary line is drawn dotted to show that its points are not included in the solution set.

*Note: When teaching graphing inequalities, it is customary to tell students to test the coordinates of a point, not on the boundary line, in the inequality. If the coordinates satisfy the inequality, the solution set consists of that point and all of its neighbors on its side of the boundary; and if not, the solution set consists of all the points on the other side of the boundary.

EQUATIONS OF LINES

From this discussion, and the problem set that follows this section, you should realize that every linear equation of the type \( ax + by + c = 0 \) (or other equivalent forms) has a line for a graph. The converse of this statement is also true. Every line has a linear equation that generates it. The next several paragraphs present procedures for finding the equation of a line.

A fundamental characteristic of a line is that its "steepness" is constant. The measure of a line's "steepness" is called its slope*, and is the ratio of the line's rise to its run as measured between any two points on the line. The rise of a line is the vertical change, and the run is the horizontal change in position that occurs in going from one point to another. By convention, rise is positive if up and negative if down, and run is positive if right and negative if left. Consider the movement in going from the point (0,2) to the point (3,0) on the line in the figure at the top of the last page. This movement is down 2 and right 3. Hence the slope is \( \frac{2}{3} \). Suppose the movement is from (3,0) to (0,2). Then the movement would be up 2 and left 3 producing a slope of \( \frac{-2}{3} \). Of course, these numbers are equal, and both are usually written as \( \frac{-2}{3} \). You could use the data in
the x-y table seen earlier in this section and the graph containing those points to verify that regardless of which two points are used, the slope of the line is a constant \(-\frac{2}{3}\).

*Note:* When introducing the concept of slope, it is helpful to use an inductive approach, wherein students really learn the meaning of slope and can picture various slopes in their minds. For example, consider the arithmetic sequence 1, 3, 5, 7, \ldots. The terms in this sequence are generated by the linear function \(y = 3x\) where \(x\) is restricted to the set of natural numbers (1, 2, 3, \ldots). To move right from one point to the next, you rise 2 for every run of 1.

To generalize the notions just discussed, consider a line determined by two points \(P(x_1, y_1)\) and \(Q(x_2, y_2)\) as pictured below.

The x-coordinate of \(V\) is \(x_2\) because \(VQ\) is a vertical line, and the y-coordinate is \(y_1\) because \(PV\) is a horizontal line. In going from \(P\) to \(Q\), the run is \(x_2 - x_1\), and the rise is \(y_2 - y_1\). As discussed in the specific example above, it doesn't make any difference whether you think of going from \(P\) to \(Q\) or \(Q\) to \(P\) in calculating the slope, but since most people read left to right, it is customary to think of moving left to right on the line. Consequently, the coordinates of the leftmost point are subtracted from those of the rightmost point. Thus, the standard formula for the slope (usually denoted by \(m\)) of a line is:
Example: Find the slope of the line below:

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

Solution: 
\[ m = \frac{4 - (-2)}{2 - (-3)} = \frac{6}{5} \]

Now suppose \( P (x,y) \) is any point on this line in the example above. Then the slope of the line calculated using \( P \) and the point \( (2,4) \), is \( \frac{y - 4}{x - 2} \). Since the slope of a line is constant \( \frac{y - 4}{x - 2} = \frac{6}{5} \), or \( y - 4 = \frac{6}{5} (x - 2) \). This latter form is preferred to the previous form because no values for \( x \) are excluded (as is \( x = 2 \) for the form on the left above). It is an example of what is called the point-slope form of a line. \((2,4) \) is a point on the line, and \( \frac{6}{5} \) is the slope of the line.

Of course, you could have used the other point, \((-3,-2)\), in this work and obtained \( y - (-2) = \frac{6}{5} (x - (-3)) \). If you take each of these equations and solve them for \( y \) they will be seen to be identical equations. For a point, \((x_1, y_1)\) and slope \( m \), the general point-slope form of the equation of a line is:

\[ y - y_1 = m (x - x_1) \]

If the known point is the y-intercept [usually denoted by \((0,b)\)], the resulting equation is:

\[ y - b = m (x - 0) \]

or 
\[ y - b = mx \]

or 
\[ y = mx + b \]

The form, \( y = mx + b \) is called the slope-intercept form of a line because \( m \) is the slope of the line, and \( b \) is the y-intercept.
Example: Use the point-slope form to find the equation of the line through the points (-3,5) and (7, -2), and then rewrite the equation in slope-intercept form.

Solution: \[ m = \frac{-2 - 5}{7 - (-3)} = \frac{-7}{10} \]

\[ y - (-2) = \frac{-7}{10} (x - 7) \]

\[ y + 2 = \frac{-7}{10} x + \frac{49}{10} \]

\[ y = \frac{-7}{10} x + \frac{29}{10} \]

Many people prefer to use the slope-intercept form \((y = mx + b)\) directly to determine the equation of a line. This technique employs the concept that if a point is on a line, its coordinates must satisfy the equation of the line. Substituting the slope \(-\frac{7}{10}\) and the point \((-3, 5)\), from the preceding example, into \(y = mx + b\) produces:

\[ 5 = -\frac{7}{10} (-3) + b \]

\[ 5 = \frac{21}{10} + b \]

\[ b = 5 - \frac{21}{10} = \frac{29}{10} \]

Thus \[ y = -\frac{7}{10} x + \frac{29}{10} \]

Now consider the points (1,3) and (5,3). The slope of the line determined by these two points is \(\frac{3 - 3}{5 - 1} = 0\). The y-intercept of the line is (0,3). Thus the equation of this line is \(y = 0x + 3\) or simply \(y = 3\). In general, if the slope of a line is 0, the line is said to be horizontal and its equation is given by \(y = k\) where \(k\) is the y coordinate of every point on the line.

Next, consider the points (1,3) and (1,5). The slope of the line determined by these two points is \(\frac{5 - 3}{1 - 1} = \infty (\text{undefined})\). There is no y-intercept, but we know the x coordinate of every point on the line is 1. Thus we write the equation of this line as \(x = 1\). In general, if the slope of a line is undefined, the line is said to be vertical and its equation is given by \(x = k\) where \(k\) is the x coordinate of every point on the line.

There are other forms for the general equation of a line. In the next sub-section, you will encounter equations in what is referred to as \textbf{standard form: }ax + by = c. Of course, one form of a linear equation can be obtained from another. For example, consider the linear equation previously found:

\[ y = -\frac{7}{10} x + \frac{29}{10} \]
Multiplying by 10 produces\[10y = -7x + 29\]

Adding $7x$ to both sides produces\[7x + 10y = 29\]which is standard form.

**Exercises:**

$x\frac{a}{a} + y\frac{b}{b} = 1$ is another form of an equation of a line called Intercept form.

1. Graph the equation $2x + 3y = 6$ and show the coordinates of the intercepts.

2. Rewrite $2x + 3 = 6$ in the intercept form by first dividing through by 6 and then simplifying.

3. Why is this form called intercept form?

**SYSTEMS OF LINEAR EQUATIONS**

You just saw that an equation of the form $ax + by = c$ (standard form) generates a line in the Cartesian Plane (or $xy$ plane). Now suppose you have two such equations and graph both of them on a single $xy$ plane. There are three possible orientations for this pair of lines. They could intersect at a single point (intersecting lines), they might not intersect at all (parallel lines), or they could intersect each other at every point (coincident lines). Many application problems in mathematics give rise to the need to solve a second order system of first degree equations. (When used in this context, the phrase second order simply means there are two variables.) One way to solve such a system of equations is to simply graph the lines (by hand or using a graphing utility such as a computer or a graphing calculator) and see if, and where, they intersect.

**Note:** One advantage of using a graphing utility is that you can inductively discover that parallel lines have the same slope and perpendicular lines have slopes which are negative reciprocals of each other (the product of the slopes equals $-1$). An investigation (which requires only basic algebra and geometry) leading to discovery of and justification for these important notions can be seen in the lab manual which accompanies this book.

This "graphical method," if done by hand, is generally not very precise, and an algebraic process that generates exact values may preferred.

You already know how to solve one equation in one unknown. To solve a linear system of two equations by the **substitution** method, first solve one equation for one of the variables in terms of the other. Then substitute this expression into the second equation. After substituting, this second equation will now have only one variable which can be solved using methods you learned previously.
Example: \[ 2x + 3y = 3 \]
\[ x + y = 2 \]
Solving the second equation for \( y \) yields \( y = 2 - x \)
Substituting in the first equation yields \[ 2x + 3(2 - x) = 3 \]
Solving for \( x \) produces \[ 2x + 6 - 3x = 3 \]
\[ -x + 6 = 3 \]
\[ -x = -3 \]
\[ x = 3 \]
Now substitute \( x = 3 \) into the equation \( y = 2 - x \), and solve for \( y \).
\[ y = 2 - x \]
\[ y = 2 - 3 \]
\[ y = -1 \]
Note: You might use your graphing calculator to show that the lines generated by the two equations intersect each other at the point \((3, -1)\).

A second method called elimination involves, as the name suggests, a process that eliminates one variable, again producing one equation in one unknown. This process makes use of the basic facts that equals added to equals are equal and equals multiplied by equals are equal. Consider the system just solved by the substitution method.

\[ 2x + 3y = 3 \]
\[ x + y = 2 \]
Multiply the second equation by \(-3\),
then add it to the first equation.
\[ 2x + 3y = 3 \]
\[ -3x - 3y = -6 \]
\[ -x = -3 \]
\[ x = 3 \]
Now multiply the second equation by \(-2\), and add it to the first equation.
\[ 2x + 3y = 3 \]
\[ -2x - 2y = -4 \]
\[ y = -1 \]
In actual practice, a combination of these two techniques is employed. When solving a second order system, it is often more efficient to use elimination to solve for one variable, and then use substitution to find the value of the second variable. When applying these techniques, you might obtain an equation such as \( 0 = 0 \). Since this equation is always true, any ordered pair that satisfies one equation will satisfy the other, and the lines represented by these equations are coincident (the same line). For example, consider the system \[ \begin{align*}
- x - y & = 3 \\
-x + y & = -3
\end{align*} \]. Adding these equations produces \( 0 = 0 \) and graphing reveals the graphs of these equations to be the same line. On the other hand, adding the equations of the system \[ \begin{align*}
- x - y & = 4 \\
-x + y & = -2
\end{align*} \] produces \( 0 = 2 \). Since this is never true, any ordered pair that satisfies one equation will not satisfy the other equation, and this system of equations is said to be inconsistent. The equations of an inconsistent second order system will graph as parallel lines.
Next, a third order system of first degree equations will be discussed. Such a system contains first degree equations having up to three variables. It is not surprising to discover that the graph of the solution set of a first degree equation in three variables consists of a set of coplanar (on the same plane) points.

Consider the equation \( x + 2y + 3z = 6 \). Choosing values for two of the variables, and then finding a corresponding value for the third variable, produces a set of ordered triples. For example, if \( x = 0 \), and \( y = 0 \), then \( z \) must equal 2. Similarly, if \( x \) and \( z \) are each 0, then \( y = 3 \), and if \( y \) and \( z \) are each 0, then \( x = 6 \). Thus, three of the ordered triples that satisfy the equation are \((0,0,2)\); \((0,3,0)\); and \((6,0,0)\). The set of all such ordered triples that satisfy the equation determines a set of coplanar points.

The figure below shows a two dimensional representation of three dimensional space. The \( x \), \( y \), and \( z \) axes are mutually perpendicular and form eight octants or regions in space. Analogous to the Quadrants formed by 2 axes in a plane, Octant I contains all points for which all three coordinates are positive numbers. The shaded "triangular region" shown is the portion of the plane determined by \( x + 2y + 3z = 6 \) that lies in Octant I. Notice that the intercepts’ coordinates are the three ordered triples found above.

Now consider a system of two first degree equations in three variables. Geometrically, you can picture the solution set for such a system as the intersection of two planes in space (analogous to the solution to a system of two first degree equations in two variables being the intersection of two lines in a plane). If the two planes are parallel planes, the solution set is empty. However, if the two planes are neither parallel nor coincident, then they will intersect in a line. The ordered triples which are solutions to the two equations in three variables will all correspond to the line of intersection of the two planes. There are, of course, an infinite number of solutions.

Now consider how three planes can intersect. Geometrically several configurations can occur with these planes, which are the graphs of first degree equations in three variables.

1. All three planes can be parallel (no common solutions).
2. Two planes are parallel and the third intersects each of these in a line (no solutions common to all three equations).
3. The three planes intersect each other in pairs forming three parallel lines.
4. Two planes intersect forming a line and the third plane intersects that line creating a single point of intersection.
5. The three planes all have the same line of intersection (infinite number of solutions).
(6) The three planes are all the same plane (infinite number of solutions).

Of these possible configurations, the fourth case which produces one solution is the only type of third order system which will be considered in this book. A sample system is solved below.

\[
\begin{align*}
x + y + z &= 2 & \text{E}_1 \\
2x - z &= -1 & \text{E}_2 \\
3y + 2z &= 0 & \text{E}_3
\end{align*}
\]

The plan is to "reduce" this system to a second order system by "eliminating" one variable. Observe that there is no "x" term in the third equation. Combine the first two equations in such a way as to eliminate \( x \) just as you did with second order systems. One way this can be accomplished is by multiplying \( \text{E}_1 \) by -2 and then adding the resulting equation to \( \text{E}_2 \).

\[
\begin{align*}
-2[x + y + z &= 2] & \Rightarrow -2x - 2y - 2z = -4 \\
2x - z &= -1 & \Rightarrow 2x - z = -1
\end{align*}
\]

\[-2y - 3z = -5\]

Now combine this equation with \( \text{E}_3 \) to form the second order system:

\[
\begin{align*}
-2y - 3z &= -5 \\
3y + 2z &= 0
\end{align*}
\]

Now solve this system using any technique you like. For Example:

\[
\begin{align*}
3[-2y - 3z &= -5] & \Rightarrow -6y - 9z = -15 \\
2[ 3y + 2z &= 0] & \Rightarrow 6y + 4z = 0
\end{align*}
\]

\[-5z = -15
\]

\[z = 3
\]

\[3y + 2(3) = 0
\]

\[3y = 6
\]

\[y = -2
\]

\[x + (-2) + 3 = 2
\]

\[x + 1 = 2
\]

\[x = 1
\]

The point (1, -2, 3) satisfies the system. Observe that a simple change in the order of a system from two to three significantly increases the complexity of the solution technique. These techniques can be used to solve even higher order systems, but more sophisticated tools, such as matrices and/or computer software programs, are generally employed.

\textbf{Question:} Which method of solving systems do you prefer: graphing, substitution, elimination, or some combination?
MATRICES

The work seen in the last subsection can be shortened by using matrices. A matrix is a rectangular array of numbers enclosed by brackets. Matrices have many applications, but at this time we will only look at how they can be employed to solve systems of linear equations. In the solution process seen in the last subsection, we added multiples of equations together to produce new equations having fewer variables. We can accomplish the same thing by first writing a matrix such that the rows of the matrix represent the equations and then combining multiples of rows so as to produce zeros (which represents eliminating variables from the equation). This process is shown below.

Consider the third order system solved previously.

\[
\begin{align*}
 x + y + z &= 2 \\
 2x - z &= -1 \\
 3y + 2z &= 0 \\
\end{align*}
\]

The matrix form for this system is

\[
\begin{bmatrix}
 1 & 1 & 2 \\
 2 & 0 & -1 \\
 0 & 3 & 2 \\
\end{bmatrix}
\]

Transforming this matrix as follows produces:

- A new matrix consisting of Row 1, -2 * Row 1 + Row 2 and Row 3 is (1).
- Next, we add row 3 to Row 2, producing (2).
- Now, add –1* Row 2 to Row 1 and –3* Row 2 to Row 3, producing (3).
- Next, dividing Row 3 by 5 produces (4).
- Finally, add –2*Row 3 to Row 1 and Row 3 to Row 2, producing (5).

\[
\begin{align*}
\text{(1)} &:
\begin{bmatrix}
 1 & 1 & 2 \\
 0 & 2 & -5 \\
 0 & 3 & 2 \\
\end{bmatrix} \\
\text{(2)} &:\begin{bmatrix}
 1 & 1 & 2 \\
 0 & 1 & -5 \\
 0 & 3 & 2 \\
\end{bmatrix} \\
\text{(3)} &:\begin{bmatrix}
 10 & 2 & 7 \\
 0 & 1 & -5 \\
 0 & 5 & 15 \\
\end{bmatrix} \\
\text{(4)} &:\begin{bmatrix}
 1 & 1 & 2 \\
 0 & 2 & -5 \\
 0 & 3 & 2 \\
\end{bmatrix} \\
\text{(5)} &:\begin{bmatrix}
 10 & 2 & 7 \\
 0 & 1 & -5 \\
 0 & 5 & 15 \\
\end{bmatrix}
\end{align*}
\]
This final matrix shows that $x = 1$, $y = -2$ and $z = 3$. Observe that we could have stopped the process with the third matrix and simply used substitution to solve the system. Matrices have many applications, others of which we will examine later in this book.

\[
\begin{bmatrix}
1 & 0 & 2 & 7 \\
0 & 1 & -5 & 7 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

Note: The parallel and perpendicular lines lab activity should be done before Problem Set 6.1.

**Problem Set 6.1**

1. Plot the following points and state the quadrant in which each point is plotted.
   
   (a) (-2, 3)    (b) (3, -4)    (c) (4, 5)    (d) (-3, -5)

2. Graph each of the following lines:
   
   (a) $2x - 3y = 12$    (b) $-4x + 5y = 20$    (c) $2x - y = 0$
   
   (d) $x = 4$    (e) $y = -5$    (f) $y = 3x - 4$

3. Graph each of the following inequalities.
   
   (a) $y > 3x - 6$    (b) $y \leq -4x + 8$    (c) $y \leq x$    (d) $y > 2$
   
   (e) $x \leq -4$    (f) $2x - 3y \leq 12$

4. Find (if it exists) the slope of the line through each of the following pairs of points.

   (a) (-2, 3) and (3, -4)    (b) (4, 5) and (-3, -5)
   
   (c) (-2, 3) and (5, 3)    (d) (4, 5) and (4, -3)

5. Write (if it exists) the slope of each of the lines in problem 2.

6. Write the equation, in standard form, of each of the lines through the points in problem 4.

7. Write the equation of (a) a horizontal line and (b) a vertical line through point (-2, 3).

8. Write the equation of a line, in point-slope form, which passes through the point (2, -1) and is parallel to the line $2x + 3y - 6 = 0$. 
9. Write the equation of a line, in slope-intercept form, which passes through the point (-3,5) and is perpendicular to the line $x - 2y + 3 = 0$.

10. The equation $2x - 3y = 12$ is in standard form. Write it in:

   (a) point-slope form (use any point you wish)
   (b) slope-intercept form
   (c) intercept form \( \frac{x}{a} + \frac{y}{b} = 1 \)

11. The equation \( ax + by + c = 0 \) is often called the general form of the equation of a line. What must be true about this general form if the line passes through the origin?

12. Solve the following systems by either carefully graphing on graph paper or by using a graphing calculator.

   (a) \[ 3x + y = 1 \quad 2x - 3y = 8 \]
   (b) \[ x + y = 2 \quad 4x + 2y = 1 \]
   (c) \[ 3x + 5y = -1 \quad 6x - 15y = 13 \]

13. Solve each of the systems in problem 12 algebraically.

14. Identify each of the following pair of lines as either coincident, parallel or intersecting.

   (a) \[ 2x - 3y = 6 \quad 3x + 2y = 10 \]
   (b) \[ 2x - 3y = 5 \quad -4x + 6y = -10 \]
   (c) \[ x - 3y = 12 \quad 4x - 12y = 24 \]

15. Use matrices to solve the third order systems below.

   (a) \[ x + y + z = 4 \quad 2x + 4y = 0 \quad 3y - z = -2 \]
   (b) \[ x + y + z = 3 \quad 2x + y - 3z = -1 \]
   (c) \[ x - 2y - 4z = 2 \]

16. **Acute Angle Problem:** One of the acute angles of a right triangle is 40° more than three times the other acute angle. Find the two angles.

17. **Digit Problem:** The sum of the digits of a two-digit number is 9. If the digits are reversed, the new number is 9 less than the original number. Find the number.

18. **Fraction Problem:** A certain fraction has a value of $\frac{1}{3}$. If the numerator is increased by 7 and the denominator is decreased by 4, the new fraction has value $\frac{3}{4}$. Find the original fraction.
19. **Wind Problem:** With the wind, an airplane travels 1120 miles in 7 hours. Coming back, against the wind, the same trip takes 8 hours. Find the wind speed.

20. **Gas Problem:** A dealer knows he pays 10 cents more per gallon for unleaded gas than he pays for regular gas. He bought 1000 gallons of regular and 3000 gallons of unleaded gas for a total cost of $3500. How much did he pay for each type of gas?

21. **Running Problem:** Irene, Jane and Kari are comparing the speeds at which they can run. The sum of their speeds is 30 mph. Irene's speed plus one-third of Jane's speed is 2 mph more than Kari's speed. Four times Jane's speed plus three times Kari's speed minus six times Irene's speed is 30 mph. How fast can each run?

22. **Tanker Problem:** An oil tanker can be emptied in 2 days (48 hours) by 3 pumps working together. The smallest and largest pumps will empty the tanker in 3 days (72 hours) while the two smaller pumps require 4 days (96 hours) to empty the tanker. Suppose both of the two larger pumps are broken. How long would it take to empty the tanker using just the smallest pump?
In the real world • • •

Overheard from one taxi driver to another:

What’s all this stupid talk about the shortest distance between two points?

In mathematics, when talking about the distance between two points, the shortest distance is implied. In this section, the distance and midpoint formulas will be developed. These notions will then be used to develop the equation of a circle. But first the important mathematical concept of function will be revisited.

As you have seen before, in mathematics, sets of ordered pairs, triples, etc. are called relations. It is frequently the case that such a set is generated by an equation. For example, first degree equations in two variables have ordered pairs which satisfy the equation and graph as a line. First degree equations in three variables have ordered triples which satisfy the equations and graph as planes. In this section only relations which are ordered pairs will be discussed.

FUNCTIONS

You have seen before that a special kind of relation between two sets, say $A$ and $B$, such that for every element in set $A$ there corresponds one and only one element in $B$ is called a function.

Examples:  

- $\{(1,2), (-1,3), (0,4)\}$ is a function
- $\{(1,2), (-1,3), (1,4)\}$ is not a function

If the non-function set of ordered pairs in the preceding example is plotted, the points $(1,2)$ and $(1,4)$ would lie on a vertical line. This observation prompts a general test, called the vertical line test, which can be used to determine if the graph of a set of ordered pairs is that of a function.

A set of ordered pairs is a function, if and only if, any vertical line passing through the graph intersects the graph in at most one point.

Consider the three graphs shown below and convince yourself that the circle graph is not that of a function, but the other two are.
Previously you learned that every linear equation whose graph is non-vertical can be written in \( y = mx + b \) form. When expressed in this manner, \( y \) is called an explicit function of \( x \), and indicated by:

\[
y = f(x) = mx + b.
\]

This is read as "\( y \) equals \( f \) of \( x \)," and \( f \) is the name of the function.

**Note:** It is common to use \( f, g, \) and \( h \); or \( F, G, \) and \( H \) as function names. It is also common to be somewhat less precise in discussing functions. For example, "Graph the function \( y = 3x - 5 \)," means, "Graph the function generated by the equation \( y = 3x - 5 \)."

In order to graph a function defined by an equation, you need to generate ordered pairs by choosing values for one variable, and then finding corresponding values for the other variable. It is common to first choose abscissa values, and then find the corresponding ordinate values.

Suppose \( f \) is a set of ordered pairs \((x, y)\) generated by \( y = f(x) = 2x - 5 \).

Since \( x \) is to be chosen first, \( x \) is called the **independent** variable. The set of values from which \( x \) can be chosen is called the **domain**. In this case, \( x \) can be any real number, so:

\[
D_f = \text{all reals} \text{ or } D_f = (\infty, \infty) \text{ using interval notation.}
\]

Since the value of \( y \) is dependent on \( x \), it is called the **dependent variable**. The set of values the dependent variable can take on is called the **range** of the function. In the example:

\[
R_f = \text{all reals} \text{ or } (\infty, \infty).
\]

Now if some domain values are selected and substituted in the function equation for \( x \), some of the ordered pairs that comprise \( f \) can be generated.

For example: If \( x = -1 \), then \( y = f(-1) = 2(-1) - 5 = -7 \).
If \( x = 2 \), then \( y = f(2) = 2(2) - 5 = -1 \).

Thus, \((-1,-7)\) and \((2,-1)\) are two elements of \( f \), and the line for \( f \) can be drawn as you have previously learned.
With regard to the second and third graphs shown on the previous page, you will work with circle graphs and their equations later. The V-shaped graph in the previous example is that of the absolute value function \( y = f(x) = |x| \). It is what is called a piecewise continuous (has no breaks) graph in that it is really a composition of two graphs.

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0.
\end{cases}
\]

To graph \( y = f(x) = |x| \), graph \( y = g(x) = x \) with \( D_g = [0, \infty) \), and \( y = h(x) = -x \) with \( D_h = (-\infty, 0) \) on the same coordinate axes.

**THE DISTANCE FORMULA**

Prior to discussing circle graphs, some additional analytic work is needed. First of all, you should realize that finding lengths of horizontal or vertical line segments is equivalent to working on a number line.

**Example:** The line segment \( AB \) in the plane is congruent to the line segment from -2 to 6 on the number line (see the figure below). The length of \( AB \) is 6 - (-2) = 8. The midpoint is 4 from either endpoint, or at (2,3).

Now consider the task of finding the length of line segment \( AB \) shown below. A horizontal segment from A, and a vertical segment from B meet at C.
The coordinates of C are (7, 2). Since horizontal and vertical lines are perpendicular, triangle ACB is a right triangle, and its side lengths satisfy the Pythagorean Theorem.

\[ AC = 7 - (-1) = 8 \]
\[ BC = 2 - (-4) = 6 \]

\[ AB^2 = AC^2 + BC^2 \]

and

\[ AB = \sqrt{AC^2 + BC^2} \]

(length is positive)

Thus for the segment in Figure 3

\[ AB = \sqrt{8^2 + 6^2} \]
\[ = \sqrt{64 + 36} \]
\[ = \sqrt{100} \]
\[ = 10 \]

To generalize the procedure above, consider the figure below.

\[ PQ = \sqrt{PV^2 + QV^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

This result is commonly called the **distance formula**, and can be used to find the distance between any two points in the plane. It need not be memorized however, since it is just another form of the Pythagorean Theorem.

**THE POINT TO LINE DISTANCE FORMULA**

Now that we have a method of finding the distance between points, it also will be useful to be able to find the distance from a point to a line. Consider the picture shown below and complete the exercise that follows to develop this useful formula.
Exercise: Draw Line PD perpendicular to Line L at D. Then find the slope of Line PD which would be the negative reciprocal of the slope of Line L. Next write the equation of Line PD.

Now find the coordinates of D (solve the two equations simultaneously).

Finally use the Distance formula to find that the distance between P and D is

\[ \frac{|ap + bq + c|}{\sqrt{a^2 + b^2}} \]

THE MIDPOINT FORMULA

Recall from geometry that medians and perpendicular bisectors both contain the midpoint of a side of a triangle. Now, reconsider the figure seen earlier, and think about how to find the midpoint of AB.
As will be proved below, the midpoint of AB will have the same x-coordinate as the midpoint (3,2) of horizontal segment AC, and the same y-coordinate as the midpoint (7,-1) of vertical segment BC. Thus, the midpoint of AB is (3,-1).

Now suppose you want to find the coordinates of the midpoint of line segment PQ shown in Figure 4. Consider the figure below which shows the same line segment with the x and y axes omitted.

Let M be the midpoint of PQ, and MD be parallel to QV. Triangles PDM and PVQ are similar by AA (both are right triangles and have angle P in common). Thus \( \frac{PM}{PQ} = \frac{PD}{PV} \).

Since M is the midpoint of PQ, \( \frac{PM}{PQ} = \frac{1}{2} \). Consequently \( \frac{PD}{PV} = \frac{1}{2} \), which means D is the midpoint of PV. The x-coordinate of D is easily found to be \( \frac{x_1 + x_2}{2} \), and since M and D lie on a vertical line, the x-coordinate of M is also \( \frac{x_1 + x_2}{2} \). Drawing a horizontal line through M and using similar reasoning will show the y-coordinate of M to be \( \frac{y_1 + y_2}{2} \).

The preceding development establishes what is called the **midpoint formula**.

The coordinates of the midpoint of the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\) are:

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]
Note: As with the distance formula, the midpoint formula need not be memorized if you understand how it was derived.

Example: The midpoint of the segment joining (-2,3) and (5,9) has coordinates:

\[ \left( \frac{-2 + 5}{2}, \frac{3 + 9}{2} \right) \text{ or } \left( \frac{3}{2}, 6 \right). \]

Both the distance formula and the midpoint formula are employed in many problems commonly encountered in mathematics including those in the problem set following this section. This section will conclude with a discussion of how to utilize the distance formula to derive the equation of a circle.

CIRCLE EQUATIONS

Recall that a circle is the set of all points in the plane equidistant from a given fixed point, called its center. Consider the circle with center (2, -3) and radius of length 5 shown below

Let P be any point on the circle. The length of PC is equal to 5 for any point P. Writing this statement using the distance formula produces:

\[ \sqrt{(x - 2)^2 + (y + 3)^2} = 5 \]

Squaring both sides yields

\[ (x - 2)^2 + (y + 3)^2 = 25 \]

The preceding equation is in standard form for the equation of a circle.

In general, if a circle has a center at (h,k) and a radius of r, its equation is

\[ (x - h)^2 + (y - k)^2 = r^2 \]
Now if you square the binomials and collect like terms you obtain:

\[
x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2
\]

or

\[
x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0
\]

and by renaming the coefficients \(x^2 + y^2 + Ax + By + C = 0\).

This latter form is the **general form** of the equation of a circle. It is often necessary to transform the general form to the standard form in order to sketch the graph. This process employs the technique of completing the square (seen in Section 2.6) as shown in the example below.

**Example:** Identify the center and radius, and then graph the following equation.

\[
x^2 + y^2 + 6x - 10y + 18 = 0
\]

Solution:

\[
(x^2 + 6x + 9) + (y^2 - 10y + 25) = -18 + 9 + 25
\]

\[
(x + 3)^2 + (y - 5)^2 = 16 = 4^2
\]

Recall that as seen previously using the vertical line test, a circle equation does not generate a function.

Consider the circle with equation \(x^2 + y^2 = 16\).

If you solve for \(y\), you obtain \(y^2 = 16 - x^2\) or \(y = \pm \sqrt{16 - x^2}\).

The equation \(y = \sqrt{16 - x^2}\), would generate a function which would graph as the "top" half of a circle, and \(y = -\sqrt{16 - x^2}\) generates another function that would produce the "bottom" half (see the figures below).
Understanding whether or not an equation generates a function is often important when using a graphing utility to produce the graph. Many technological tools require that the dependent variable be explicitly expressed as a function of the independent variable.

For example, in order to have a graphing calculator, or a computer, graph a circle equation, such as $x^2 + y^2 = 16$, it must be programmed to graph the two semicircle functions, $y = \sqrt{16 - x^2}$ and $y = -\sqrt{16 - x^2}$ on the same coordinate axes.

**Note:** In a later chapter you will use parameterization to graph non functions.

The equations of circle graphs are second degree equations in two variables. Other similar equations and their graphs will be discussed later in this chapter.

**Problem Set 6.2**

1. Identify each of the following sets as a function or a non-function.
   (a) $\{ (1, 2), (2, -3), (3, 4), (-2, 1), (0, 2)\}$
   (b) $\{ (1, 2), (-3, 2), (4, 1), (1, -3), (0, 2)\}$

2. Identify each of the following graphs as being that of a function or a non-function. Explain.
   (a) ![Graph A](image)
   (b) ![Graph B](image)
   (c) ![Graph C](image)

3. Given $y = f(x) = 3x - 4$. Find:
   (a) $f(0)$
   (b) $f(-2)$
   (c) $f(w)$
   (d) $f(2x + 1)$
   (e) $x$ such that $y = 7$

4. Given $f(x) = 5x^3 - 2x^2 + 3x - 1$. Find:
   (a) $f(0)$
   (b) $f(-2)$
5. Graph the following pairs of points and find (i) the distance between each pair of points and (ii) the coordinates of the midpoint of the line segment joining each pair of points.

(a) (-1, 2) and (3, -5)  
(b) (2, 5) and (2, -7)

6. Determine the coordinates of point A such that the point (2, 3) is the midpoint between point A and the point (6, 10).

7. Find the distance from the point (1,2) to the line 3x - y + 3 = 0

8. Find the distance between the parallel lines 2x - 3y + 3 = 0 and 2x - 3y + 12 = 0.

9. Write the equation of the circle:

(a) with a center (0, 0) and a radius of 5 units.
(b) with a center (-1, 2) and a radius of 3 units.
(c) with a center (2, -3) and a diameter length of 8 units.
(d) with a diameter having endpoints (-5, 6) and (19, -4).
(e) with a radius of 5 and tangent to the positive x and y axes.

10. Identify the center and radius and graph:

(a) $3x^2 + 3y^2 - 18 = 0$
(b) $x^2 + y^2 - 6x + 8y = 0$
(c) $4x^2 + 4y^2 - 4x + 12y + 1 = 0$

11. Use the figure below to prove that an angle inscribed in a semicircle is a right angle. (Prove $m \angle APB = 90^\circ$.

12. Recall that if a graph passes through a point, then the coordinates of that point must satisfy the equation of the graph. Find the equation of the circle which passes through the points (-2, 0), (5, 7) and (6, 0). [Hint: Substitute the coordinates in the general form of a circle equation and solve the resulting system of equations for A, B and C.]
6.3 COORDINATE PROOFS

Coordinatizing figures provides a simple way to discover and/or prove notions about their features. For example, we know that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. (since a right angle is inscribed in a semicircle) Shown below is a coordinate proof of this fact.

First place a general right triangle in the Cartesian plane and label its vertices. The algebra involved in the proof is simplified by a thoughtful placing of the figure in the plane.

![coordinate proof diagram]

M has coordinates \( \left( \frac{a}{2}, \frac{b}{2} \right) \) by the midpoint formula. Since CM and BM are equal (because M is the midpoint), all we need to show is that AM equals either one of them. We will show AM = CM.

\[
AM = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{1}{2} \sqrt{a^2 + b^2} \quad \text{by the distance formula.}
\]

\[
CM = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2} - b\right)^2} = \frac{1}{2} \sqrt{a^2 + b^2} \quad \text{by the distance formula.}
\]

Hence AM = CM and the proof is complete.

It is certainly the case that the choice of location for the figure significantly affects the amount of algebra necessary in the proof.
**Exercise:** Use the figure below to prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.

**Hint:** You will need to use the fact that the slopes of AB and AC are negative reciprocals since A is a right angle.

Hopefully the preceding exercise convinced you to choose wisely when using coordinate proofs (and in your other endeavors in life as well). As a second example problem, we will prove that the diagonals of a parallelogram bisect each other.
Note: Since DC is parallel to AB which has slope 0, the y-coordinate of points C and D are represented using the same letter.

We will show that the midpoints of both diagonals are the same point. M₁ on AC is \( \left( \frac{b}{2}, \frac{c}{2} \right) \) and M₂ on DB is \( \left( \frac{a+d}{2}, \frac{c}{2} \right) \).

Since AD \parallel BC, we have \( \frac{c}{d} = \frac{c}{b-a} \Rightarrow d = b - a \Rightarrow b = a + d \)
and hence M₁ = M₂.

As a final example of this technique, we will prove the familiar Midline Theorem, seen in Euclidean geometry.

The Segment Joining the Midpoints of Two Sides of a Triangle Is Parallel To and Half the Third Side of the Triangle

Clearly, DE \parallel AB because both have a slope of 0.

\[
DE = \frac{a + b}{2} - \frac{b}{2} = \frac{a}{2}.
\]
So \(DE = \) half of AB.

Perhaps you noticed that the coordinate proof for the Midline Theorem was easier than previous proofs seen for this theorem. As in life, in math when faced with a problem, it is wise to decide in advance, what to do in order to solve the problem. Deciding what to do includes deciding what tools to use. For this Midline Theorem, as well as many other proofs, we could use a synthetic proof, a transformational proof, an analytic geometry proof or a vector proof (as we will see in a later chapter). It is nice to have an extensive set of tools from which to choose. Nothing beats having the right tool for the job.

Problem Set 6.3
1. Prove the diagonals of a rectangle are equal.

2. Prove the diagonals of a rhombus are perpendicular.

3. Prove that ABCD shown below is a parallelogram by:
   (a) showing both pair of opposite sides are parallel.
   (b) showing both pair of opposite sides are equal.

   ![Diagram of ABCD]

4. Prove that a parallelogram is formed by joining the midpoints of consecutive sides of any quadrilateral.

5. Prove that a rhombus is formed by joining the midpoints of consecutive sides of any rectangle.

6. Prove that the perpendicular bisectors of the sides of a triangle are concurrent in the circumcenter of the triangle.

7. Prove that the angle bisectors of a triangle are concurrent in the incenter of the triangle.

8. Prove that the altitudes of a triangle are concurrent.

9. Prove that the medians of a triangle are concurrent. Recall that the intersection of the medians is the center of gravity (called the centroid) of the triangle. Find the coordinates for this point.

10. Prove that the area of a kite equals half the product of the diagonals.

11. Prove that a parallelogram is formed by lines drawn through the trisection points closest to each vertex point of any quadrilateral. The center of gravity (centroid) for this parallelogram is also the centroid for the quadrilateral. Find the coordinates for this point.
A second degree equation of the form $y = ax^2 + bx + c$, $a \neq 0$ is also called a quadratic function. Specifically, it is second degree in one variable (x in this example) and first degree in the other (y). As shown in the examples below, generating and plotting points on the standard xy axis (x-horizontal and y-vertical) produces "U" shaped graphs called \textbf{parabolas} which open up if $a > 0$, and down if $a < 0$. If you have a graphing calculator or a computer available, generate some graphs and inductively convince yourself this is true.

\textbf{Examples:}

\[ y = x^2 - x - 2 \quad \text{and} \quad y = -x^2 + x + 2 \]

\textbf{Note:} In the left figure, notice that the x-intercepts are the points (-1,0) and (2,0). Since the y-coordinate of an x-intercept is always 0, the significant part of the ordered pair is the x-coordinate. X-coordinate numbers such as -1 and 2 in the figures are often referred to as \textbf{roots}. (Think of a tree. Where a tree's trunk goes underground it becomes the roots of the tree.) The roots of a graph are where the graph "goes under the horizontal axis." It is common practice to simply show roots beside the x-intercept points on a graph as is in the right figure. A similar procedure is employed for indicating the y-intercept. The y-intercept in the left figure is designated by (0, -2), and in the right figure simply by a 2.

Plotting more points will reveal that the graph in the left figure above will have a minimum (lowest) point and that the graph in the right figure will have a maximum (highest) point. (If you have a graphing utility, use the trace function to inductively convince yourself this is true.) Such a point on a parabola is called the parabola's \textbf{vertex}. Maximum and minimum values have extensive application in the real world. Every business would like to both minimize costs and maximize profits. One of the jobs of a computer systems analyst is to utilize mathematical models, such as quadratic equations, and develop computer programs which find maximums and/or minimums.

The figure below shows a basic parabola generated by $y = x^2$.
Coordinates of 5 points on the graph are shown. Several observations concerning the graph can be made. First, since $x$ is squared, you know $x^2 \geq 0$ for all $x$. Thus the minimum value for $y$ is 0 and $(0,0)$ is the vertex of the parabola. Second, since squaring a negative number such as -2 produces the same value as squaring its additive inverse (2 in this instance), the graph is symmetric about the y-axis. If you imagine folding one half of the graph over the y-axis, it would match up exactly with the other half. Finally, even though the graph gets steeper as it moves away from the origin, it never becomes vertical and for each $x$ value there corresponds exactly one $y$ value. Consequently, the graph of $y = x^2$ is that of a function. The domain of this function is all real numbers $(-\infty, \infty)$. The range would be $[0, \infty)$ since $y$ takes on all non-negative values. Now some modifications of this basic parabola will be considered.

**Exercise:** For each of the following equations, view the graph on a graphing utility or plot a few points (as shown for $y = x^2$ above), sketch the graph, and then discuss how the graph differs from the graph of $y = x^2$.

\[
\begin{align*}
y &= x^2 + 1 \\
y &= x^2 - 3 \\
y &= 5x^2 \\
y &= \frac{1}{5}x^2 \\
y &= -3x^2 \\
y &= (x + 2)^2 \\
y &= (x - 1)^2 \\
y &= (x + 2)^2 - 3
\end{align*}
\]

You should have discovered the following *transformations* on the basic parabola.
For \( y = x^2 + 1 \), the graph shifts up 1 units.
For \( y = x^2 - 3 \), the graph shifts down 3 units.
For \( y = 5x^2 \), the graph is steeper.
For \( y = \frac{1}{2}x^2 \), the graph is flatter.

For \( y = -3x^2 \), the graph is flipped over and is steeper.
For \( y = (x + 2)^2 \), the graph shifts to the left 2 units.
For \( y = (x - 1)^2 \), the graph shifts to the right 1 unit.
For \( y = (x + 2)^2 - 3 \), the graph shifts to the left 2 units and down 3 units.

*Note: Transformations will be discussed in detail later in this chapter.

Now consider the general quadratic function: \( y = ax^2 + bx + c \)

Factor \( a \) from the "x" terms \( y = a\left(x^2 + \frac{b}{a}x\right) + c \)

Complete the square on x
\[
y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}
\]
\[
y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}
\]

Questions:
1. Can \( \left(x + \frac{b}{2a}\right)^2 \) ever be negative?
2. If \( \left(x + \frac{b}{2a}\right)^2 = 0 \), what must x equal?
3. For this value of x, what must y equal?
4. If \( a > 0 \) and \( x \neq -\frac{b}{2a} \), is y larger or smaller than \( \frac{4ac - b^2}{4a} \) ?
5. If \( a > 0 \), does the parabola open up or down?
6. Does this parabola have a maximum or a minimum point?
7. What are the coordinates of this point?

You should have come to the conclusion that if \( a > 0 \), the parabola opens up and has a minimum point at \( \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) \). In a similar manner, you can reason that if \( a < 0 \), then the parabola opens down, and \( \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) \) is the maximum point on the graph. Notice that the formula (called the **Vertex Formula**) is the same for each of these points.
Now recall that the graph of the basic parabola, \( y = x^2 \), was seen to be symmetric about the \( y \)-axis. Actually every parabola which opens vertically has a similar type of symmetry. Such parabolas are symmetric about a vertical line (called the \textbf{axis of symmetry}) which passes through the vertex of the parabola.

**Exercise:** Draw the axis of symmetry on the parabola below, and find some corresponding pairs of points on the parabola to verify the preceding statement. \([\text{Hint:} \text{Use the x-intercepts as one pair.}]\) State the equation for the axis of symmetry (remember it is a vertical line).

![Parabola Diagram]

You should have found the equation of the axis of symmetry to be \( x = \frac{1}{2} \). You also might have observed that the axis of symmetry is the perpendicular bisector of the line segment joining the roots. To see why this is so, consider the general quadratic equation \( y = ax^2 + bx + c \). Setting \( y = 0 \), produces \( ax^2 + bx + c = 0 \), hence the roots are:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

To find the midpoint between these roots recall that you simply add them together and divide by two. This will produce \( \frac{-b}{2a} \) which is the \( x \)-coordinate of the vertex. This discovery eliminates the need for a formula for the vertex of a parabola.

\textbf{Note:} The roots may be found more easily by factoring. For example consider \( y = x^2 - 3x - 10 = (x-5)(x+2) \). The roots are 5 and -2.

**The \( x \)-coordinate of the vertex of a parabola is the average of the roots (even if they are imaginary). The \( y \)-coordinate can be found by substituting the \( x \)-coordinate in the equation of the parabola, and solving for \( y \).**

**Exercise:** Find the coordinates of the vertex of the parabola in the preceding exercise using the method just discussed. Then use the vertex formula to verify that each method produces the same point.
In the last section the pervasive mathematical notion of a function was discussed. You have seen that a quadratic equation of the form $y = ax^2 + bx + c$ graphs as a parabola that opens either up or down. By the vertical line test, such a graph represents a function.

**Example:** Consider $s = g(t) = t^2 - 2t - 3$

In set notation, function $g = \{(t,s) \mid s = t^2 - 2t - 3\}$

$t$ is the independent variable, and $s$ is the dependent variable.

$D_g = \text{all reals or } (-\infty, \infty)$

$R_g = \text{all reals } \geq -4 \text{ or } [-4, \infty)$

**Note:** The range is found by remembering that the graph of this function opens up; hence the dependent variable values will all be greater than or equal to the value of $s$ at the vertex.

**Exercise:** Verify the range for the example above.

A parabola can also open horizontally. Consider the equation $x = y^2$ which graphs as a parabola opening to the right as pictured on the left below. Clearly, by the vertical line test, this graph is not that of a function. For each $x > 0$, there are two corresponding $y$ values. For example, the points $(4,2)$ and $(4,-2)$ both belong to the graph. Even though the equation does not produce a function, $x$ and $y$ are still related, and this type of equation generates a relation.

Now, if you solve for $y$ you obtain $y = \pm \sqrt{x}$. The function, $y = f(x) = \sqrt{x}$ produces the upper half of the parabola shown on the right below. In this case, $f$ is called the square root function.

We now return to quadratic functions for which the graph is a parabola with a vertical axis of symmetry. Earlier, perhaps you thought about the following question. When does a parabola fail to have real roots, and how would this effect the graph of the parabola? You will find the answer to this question by working through the following exercises.
Exercises:
In the quadratic formula, the expression which occurs under the radical, $b^2-4ac$, is commonly called the **discriminant** because it discriminates the nature of the roots. Look back at the first example in this section, and calculate the value of the discriminant. (You should find that in this case $b^2 - 4ac > 0$.)

Next, consider the equation $y = x^2 - 6x + 9$. Show that the discriminant value is **0**. Show that the roots of the parabola are the same number. Plot several points and make a conjecture as to a special feature of a parabola whose roots are the same number.

Finally, verify that the discriminant for the parabola having equation $y = x^2 + 6x + 13$ is less than zero, and hence the roots are imaginary. Plot as many points as needed to produce a good picture of the graph.

The results of these activities are summarized below.

- If $b^2 - 4ac > 0$, the parabola has two real distinct roots (i.e. it crosses the horizontal axis at two different places).
- If $b^2 - 4ac = 0$, the parabola has a double root, and is tangent to the horizontal axis at its vertex.
If $b^2 - 4ac < 0$, the parabola has no real roots, and hence does not cross the horizontal axis.

One of the many applications of parabolas occurs in maximization/minimization problems for which the least (minimum) or greatest (maximum) value for some variable is to be found. To illustrate this important notion consider the following statement.

**The maximum area for a set of rectangles, all having the same perimeter, occurs when the rectangle is a square.**

The truth of this statement will be illustrated for a set of rectangles each having a perimeter of 20 cm.

Let $x$ represent one dimension of a rectangle with perimeter 20 cm shown on the left below. Then $10 - x$ would represent the other dimension.

![Diagram](image)

The area, $A$, of this rectangle is $A = x(10 - x) = -x^2 + 10x$.

The graph of this quadratic equation is shown on the right above.

**Note:** The vertical axis is named $A$ to correspond with area in this problem.

Notice that the graph has a maximum point when $x = \frac{0 + 10}{2} = 5$, which means the rectangle with maximum area is a 5 cm by 5 cm square.

This procedure can be generalized for a set of rectangles all having a perimeter of $P$.

The maximum area would be enclosed by a square having side length, $\frac{P}{4}$.

**INEQUALITIES**

Quadratic inequalities are of the form $ax^2 + bx + c < (\text{or} \leq \text{or} > \text{or} \geq) 0$. This type of inequality can be easily solved by thinking of the related graph.

Consider the problem $x^2 - 3x < 10$. First rewrite it in standard form as $x^2 - 3x - 10 < 0$. 


Next visualize the graph of \( y = x^2 - 3x - 10 \) as being a parabola opening up with x-intercepts of 5 and -2. Now \( y \) will be negative between the x-intercepts and hence the solution set is the open interval \((-2,5)\).

**Problem Set 6.4**

1. For each of the following equations (i) state how the graph will compare to that of \( y = x^2 \) and (ii) sketch a picture of the graph.
   (a) \( y = x^2 - 2 \)  
   (b) \( y = x^2 + 4 \)  
   (c) \( y = 3x^2 \)  
   (d) \( y = \frac{1}{3} x^2 \)  
   (e) \( y = (x - 2)^2 \)  
   (f) \( y = (x + 3)^2 \)  
   (g) \( y = - x^2 \)  
   (h) \( y = (x - 2)^2 + 3 \)

2. Write the equation of a parabola which would be each of the following modifications of the graph of \( y = x^2 \).
   (a) shifted right 5  
   (b) shifted down 4  
   (c) steepened by a factor of 6  
   (d) shifted left 3 and up 5  
   (e) flipped over the x axis and then shifted right 2 and down 4

3. Find the coordinates of the vertex for:
   (a) \( y = x^2 - 3x - 10 \)  
   (b) \( y = - 6x^2 + 11x + 10 \)  
   (c) \( y = x^2 - 4x - 1 \)

4. Graph each of the following showing the intercepts, the vertex, and the axis of symmetry.
   (a) \( y = - x^2 + 3x + 10 \)  
   (b) \( y = 6x^2 - 11x - 10 \)

5. Write the equation of a parabola which:
   (a) has intercepts (-2, 0), (0, 3) and (4, 0)  
   (b) passes through the origin and has vertex (3, -8)  
   *(c) Passes through (-1, 0), (1, -12) and (3, 24) *

6. **Rug Problem:** A rectangular rug is 3 feet longer than it is wide and has an area of 154 square feet. Find the perimeter of the rug.

7. **Falling Body Problem:** An object near the surface of the earth will fall toward the earth according to the equation \( S = - 16 t^2 + h \) where \( h \) is the initial height of the object and \( S \) is the height above the earth at any time \( t \) measured in seconds. Suppose an object is dropped from the top of a 240 ft. tall building. How long will it take the object to reach the ground?
8. **Farmer Problem:** Farmer F. has 600 meters of fence with which to fence in a rectangular field. What are the dimensions of the largest field that can be enclosed?

9. **Farmer Problem II:** Rework problem 8 if the field is to be enclosed along side a long straight road which already has a fence separating the road from the field.

10. **Box Problem:** An open rectangular box is to be made from a 8 foot by 15 foot piece of cardboard by cutting squares from each corner and turning up the sides. Find the dimensions of the box with maximum volume. [**Note:** A graphing calculator or calculus is needed for this problem.]

11. Write the solution sets for each of the following in interval notation.
   (a) $x^2 \geq 3x + 10$  (b) $6x^2 - 11x < 10$
6.5 INTRODUCTION TO THE CONICS

The parabola and the circle graphs discussed earlier belong to the general class of curves called conics. The name comes from the double cone used to demonstrate the various curves of this "family."

A circle is obtained by passing a horizontal plane through the double cone (it will be a point if the plane passes through the vertex). An ellipse is obtained if the cutting plane is tilted at an angle less than \( \theta \) (again it is a point if the plane passes through the vertex). A parabola is obtained if the cutting plane is tilted at an angle equal to \( \theta \) (a line is formed if it passes through the vertex). A hyperbola is obtained if the cutting plane is vertical (intersecting lines result if the plane passes through the vertex).

The conics have many applications in modeling real world phenomena. Merry-Go-Rounds and wheels go around in circles. Satellite antennas, radar dishes, flashlight reflectors, etc. are parabolic. Planets revolve about the sun in elliptical orbits. Some comets have orbits which are hyperbolic or elliptical.

Now, you already know that a circle is defined as the **locus** (set) of points in a plane that are equidistant from a given fixed point called the **center**. You also know that the distance of each point of the circle from the center is called the **radius**. The other conics can be defined in a similar manner.

THE PARABOLA

A **parabola** is the locus of points in a plane such that the distances of each point in the set from a given fixed point called the **focus** and a given line called the **directrix** are equal. Consider the graph below where the focal point is \((0,p)\) and the directrix is \(y = -p\). Observe that the vertex of the parabola is midway between the focus and the directrix.

Let \( P \) be a point on the parabola. Then \( PF = PD \) and hence

- squaring both sides produces: \( \sqrt{x^2 + (y - p)^2} = y + p \)
- expanding produces: \( x^2 + (y - p)^2 = (y + p)^2 \)
- collecting terms produces: \( x^2 = 4py \)

THE ELLIPSE
An ellipse is the locus of points in a plane such that the sum of the distances of each point on the ellipse from two given fixed points called the foci is a constant. Consider the graph below where the foci are (c,0) and (-c,0). Observe that if P is at (a,0), then PF + PG = 2a and if P is at (0,b), then \(a^2 - c^2 = b^2\).

Let P be a point on the ellipse. Then \(PF + PG = 2a\)
and hence \(\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a\)
separate the radicals \(\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}\)
square both sides \((x+c)^2+y^2 = 4a^2 - 4a \sqrt{(x-c)^2+y^2} + (x-c)^2+y^2\)
isolate the radical \(4a \sqrt{(x-c)^2+y^2} = 4a^2 + (x-c)^2 - (x+c)^2\)
expand \(4a \sqrt{(x-c)^2+y^2} = 4a^2 + x^2 - 2cx + c^2 -x^2 - 2cx - c^2\)
collect terms \(4a \sqrt{(x-c)^2+y^2} = 4a^2 - 4cx\)
divide by 4 \(a \sqrt{(x-c)^2+y^2} = a^2 - cx\)
square both sides \(a^2(x-c)^2+ay^2 = (a^2 - cx)^2\)
expand \(a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2\)
collect terms \(a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2\)
factor \((a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)\)
sub in \(b^2\) for \(a^2 - c^2\) \(b^2x^2 + a^2y^2 = a^2b^2\)
divide by \(a^2b^2\) to get \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

Now in the ellipse graph shown above, it looks like \(a > b\). When this is true the horizontal diameter, which has length \(2a\), of the ellipse is called the major axis and the vertical diameter, which has length \(2b\), is called the minor axis. If \(b > a\), then these labels are reversed. Note: The foci always lie on the major axis.

THE ECCENTRICITY OF AN ELLIPSE
**Exercise:** One nice way to draw an ellipse is to first put two thumbtacks into a piece of cardboard and put a loose loop of string around the thumbtacks. Then with a pencil, pull the string tight and trace around keeping the string tight. Do this several times varying the distances between the thumbtacks. You should discover that as the thumbtacks become further apart the ellipse becomes more elongated.

Quantification of this elongation is called **eccentricity** which is defined as the ratio \( e = \frac{c}{a} \). Clearly \( 0 < e < 1 \) since \( c < a \). This notion also allows you to make a connection between circle and ellipse graphs. As \( c \) approaches 0, an ellipse becomes less elongated and looks more like a circle. So we could say that a **circle is an ellipse with** \( e = 0 \). This notion makes sense since if \( c = 0 \), then \( a = b \) and our ellipse equation becomes a circle equation!

**THE HYPERBOLA**

A **hyperbola** is the locus of points in a plane such that the difference of the distances of each point on the ellipse from two given fixed points called the foci is a constant. Consider the graph below where the foci are \((c,0)\) and \((-c,0)\) and the vertices are at \((-a,0)\) and \((a,0)\). Observe that if \( P \) is at \((a,0)\) then \( PF - PG = 2a \). Also observe that there are two parts called **branches** for the graph.

**Exercise:** Repeat the process seen for the ellipse to show that the equation for this hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). This time however, let \( b^2 = c^2 - a^2 \).

A hyperbola also has a measure of elongation again called eccentricity and defined the same as for an ellipse as \( e = c/a \). For a hyperbola observe that \( e > 1 \). With some experimenting you could also discover that if the "x" term is positive, as in our equation...
in the preceding exercise, then the hyperbola opens horizontally; but if the "y" term is positive, then the hyperbola opens vertically.

THE ASYMPTOTES OF A HYPERBOLA

Consider the hyperbola equation developed above and solve for y.

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow a^2y^2 = b^2x^2 - a^2b^2 = b^2(x^2 - a^2)
\]

\[
\Rightarrow y^2 = \frac{b^2}{a^2}(x^2 - a^2) \Rightarrow y = \pm\frac{b}{a}\sqrt{x^2 - a^2}
\]

Observe that as \(x \to \pm \infty\), the \(a^2\) term becomes insignificant and thus \(y \to \pm\frac{b}{a}\) x and thus \(y = \pm\frac{b}{a}\) x are the equations of what are referred to as the asymptotes of the hyperbola. An asymptote is a line a graph approaches.

Shown below is our general hyperbola along with its asymptotes.

![Hyperbola with Asymptotes](image.png)

Observe that the asymptotes intersect midway between the vertices of the hyperbola. As we just saw in the equations for the asymptotes, the slopes are \(\pm\frac{b}{a}\). A useful technique for graphing hyperbolas, is first to draw a rectangle of width \(2a\) and height \(2b\) and diagonals intersecting at the center of the hyperbola, as shown below.

![Rectangle and Diagonals](image.png)

Extending the diagonals forms the asymptotes. You might also note that the length of the diagonal is the focal distance \(2c\).

FOCUS-DIRECTRIX DEFINITIONS OF THE CONICS
Recall that the parabola was defined as the locus of points in a plane such that the
distances of each point of the parabola from a given fixed point called the focus and a
given line called the directrix are equal. Now the ellipse and the hyperbola can also be
defined in terms of a focus and a directrix. Actually since these two conics have two
foci, they will also have two directrices. For the ellipse, the distance of each point on the
curve from a focus is less than the distance from the corresponding directrix, but for the
hyperbola it is greater. The general notions concerning conics allow us to make a
unifying definition for the conics other than the circle:

With a focal point, F, a directrix line, L, and a point on the conic, P, form the ratio of
distances PF to PD where PD is the distance from P to line L. Then:

If this ratio equals 1, the conic is a parabola as seen previously.
If this ratio is less than 1, the conic is an ellipse.
If the ratio is greater than 1, the conic is a hyperbola.

Although this distinctive unifying feature of the conics is interesting we will not spend
any time with it in this book.

**CONIC EQUATIONS**

Now all the conics can be represented by the general second degree equation:

\[ ax^2 + bxy + cy^2 + dx + ey + f = 0 \]

If \( b \neq 0 \) the conic is "rotated". For example, the graph of \( xy = 1 \) is a rotated hyperbola
and is shown below. The angle of rotation for this graph is \( 45^\circ \) counterclockwise. In
general the angle of rotation can be found using trigonometry. We will not deal with
rotations in this book.

\[ \begin{align*}
  y \\
  x
\end{align*} \quad xy = 1 \]

If \( b = 0 \), then the equation can be written in standard form by the process of completing
the square as previously seen for the circle and the parabola.

If \( a = c \), then the conic is a circle (or possibly a point or imaginary).
If \(a\) or \(c\) but not both \(= 0\) then the graph of the conic is a parabola.

**Exercise:** Consider \(2x^2 + 2y^2 - 12x + 8y + 17 = 0\). First divide through by 2 and then complete the square for both \(x\) and \(y\) to obtain \((x - 3)^2 + (y + 2)^2 = \frac{9}{2}\) which is the equation of a circle with a center at \((3,-2)\) and a radius of \(\frac{3}{\sqrt{2}}\).

If \(a\) or \(c\) but not both \(= 0\) then the graph of the conic is a parabola.

**Exercise:** Complete the square and describe the graph of \(2y^2 + x - 4y = 0\).

If \(a \neq c\), but they have the same sign, the conic is an ellipse (or a point or imaginary).

**Exercise:** Complete the square and describe the graph of \(4x^2 + 9y^2 + 8x - 36y + 4 = 0\).

Finally, if \(a\) and \(c\) have opposite signs, the conic is a hyperbola.
Exercise: Complete the square and describe the graph of

$$4x^2 - 9y^2 + 8x + 36y - 68 = 0.$$
For Problems 1 - 7, identify the conic and then, if possible, sketch the graph showing the following: for the circle the center and the radius; for the parabola the vertex and the focus; for the ellipse the center and the endpoints of both the major and minor diameters; and for the hyperbola the vertices and the equations of the asymptotes.

1. \( x^2 + y^2 + 6x - 4y - 12 = 0 \)
2. \( 25x^2 + 9y^2 - 200x + 18y + 184 = 0 \)
3. \( 2y^2 + x - 12y + 10 = 0 \)
4. \( 9x^2 - 4y^2 + 90x + 32y + 197 = 0 \)
5. \( 9x^2 + 4y^2 - 18x + 16y + 25 = 0 \)
6. \( 9x^2 - 4y^2 - 18x - 16y - 7 = 0 \)
7. \( x^2 + y^2 + 4x - 6y + 14 = 0 \)

For Problems 8 - 12, find the equation of the conic.

8. A parabola with a focus at \((2, 3)\) and a directrix \(x = 4\).
9. A locus of points \(P\) such that \(PD = 2 PF\), where \(PD\) is the distance from \(P\) to the line \(x = 4\) and \(PF\) is the distance from \(P\) to the focal point \((-2, 0)\).
10. An ellipse such that the endpoints of the major axis are \((-2, 6)\) and \((-2, 0)\) and the eccentricity is \(\frac{2}{3}\).
11. A hyperbola with asymptotes \(y = 3x + 12\) and \(y = -3x - 12\) and containing the point \((-2, 3\sqrt{5})\).
12. A parabola with a focus at \((2, 2)\) and a directrix \(y = -x\).

For Problems 13 - 15, solve the nonlinear system and graph.

13. \( \begin{cases} y - x^2 + 4 = 0 \\ x - 2y + 2 = 0 \end{cases} \)

14. \( \begin{cases} x^2 + 4y^2 = 100 \\ x^2 + y^2 = 52 \end{cases} \)

15. \( \begin{cases} x^2 + y^2 = 8 \\ xy = 4 \end{cases} \)
6.6 THE ALGEBRA OF FUNCTIONS

Previously you have studied linear and quadratic functions and have seen the graphs of the square root function \( f(x) = \sqrt{x} \). One simple way that new functions can be created from old functions is by addition, subtraction, multiplication and division. For example the linear function \( f(x) = 2x - 3 \) can be created by adding the function \( g(x) = x \) to itself and then subtracting the function \( h(x) = 3 \). Performing these four operations on functions is done in the same manner as when doing arithmetic. In the process of creating new functions, denominators and radicals may occur. You already know that the domain for a function \( f(x) \) consists of the set of permitted values for \( x \). There are really just two rules for determining the domain of a function (or a relation) to graph in the real plane.

1. Do not let a denominator (if there is one) become 0.
2. Do not let even index \( \sqrt{}, \sqrt[4]{}, \text{etc.} \) radicands become negative.

**Examples:** For \( f(x) = \frac{2}{x^2 - x} \) the domain is all reals except 0 and 1.

For \( g(x) = \sqrt{2x - 3} \) the domain is all reals \( x \geq 3/2 \).

The only new thing you must realize when combining functions is that **you may only combine two functions over their common domain.**

**Example:** Let \( f(x) = 2x - 3 \) and \( g(x) = \sqrt{x} \).

Now it is easy to see that \( D_{f+g} = [0, \infty) \).

However \( D_{g^2} = [0, \infty) \) even though \( g^2(x) = x \).

Another way to create new functions is called **composition.** A composite function is formed by taking a function of a function.

**Example:** Suppose that \( z = f(y) = 2y - 3 \) and \( y = g(x) = \sqrt{x} \).

Then \( z = f(y) = 2 \cdot 3 = 2g(x) - 3 = 2\sqrt{x} - 3 \).

In the example above, \( z = f(y) = f[g(x)] \) which is sometimes written as \( (f \circ g)(x) \). This prompts the question, "Could we find \( (g \circ f)(x) \)?" The answer is yes.

Since \( f(y) = 2y - 3 \), \( f(x) = 2x - 3 \).

\( (g \circ f)(x) = g[f(x)] = \sqrt{2x - 3} \).

An important observation should be made here. In general, the reverse compositions of two functions are not equal. In the work shown above, observe that \( f[g(x)] \neq g[f(x)] \).

Now consider two general functions \( f(x) \) and \( g(x) \) and let \( y = (f \circ g)(x) = f[g(x)] \). To obtain ordered pairs for this composite function, observe that you would chose \( x \)-values, put them in the "g" function, get range values for the "g" function, substitute these values in the "f" function and finally get the range values for the composite function. In other
words it appears that the domain for the composite function \( f[g(x)] \) consists of the range values for \( g(x) \). In actuality the domain may be reduced to a subset of \( R_g \) because of the nature of the "f" function.

Example: Let \( f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{x} \). \( f[g(x)] = \sqrt{\frac{1}{x}} \)

\( R_g = \text{all reals} \neq 0 \) but negative values \( \not\in D_{f \circ g} \)

**TRANSFORMATIONS**

Previously you were introduced to the concept of transformations. In this subsection we first will make a list of some commonly used basic algebraic functions and their graphs. We will then examine how these *library function* graphs can be modified, i.e. *transformed*. This work will empower you with the ability to easily graph many functions. Even if you are using technology, such as a graphing calculator or a computer, to do the graphing, if is very helpful to know in advance what the general shape of some graph should look like. This helps to guard against gross errors. Additionly, because one of the most important applications of elementary functions is that of modeling (representing) a data set with an equation, having the ability to recognize graph shapes is very useful.

Shown below is a set of some common algebraic library functions and their graphs.

<table>
<thead>
<tr>
<th>The Constant Function</th>
<th>The Linear Function</th>
<th>The Absolute Value Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = k )</td>
<td>( f(x) = x )</td>
<td>( f(x) =</td>
</tr>
</tbody>
</table>

\[ (0,k) \]

The Square Function The Sq. Root Function The Cube Function

| \( f(x) = x^2 \) | \( f(x) = \sqrt{x} \) | \( f(x) = x^3 \) |

\[ \]

The Cube Root Function The Reciprocal Function The Step Function
Either by hand or with a grapher, first graph \( g(x) = |x + 2| \) and \( h(x) = |x - 3| \).
You should have discovered that the graph of \( g(x) \) translated (shifted) horizontally left 2 units the library absolute value function graph (\( f(x) = |x| \)) and that of \( h(x) \) translated \( f(x) \) right 3 units. \((1,1) \in f \Rightarrow (-1,1) \in g \) and \((4,1) \in h)\]

**In general** \( f(x - h) \) translates the graph of \( f(x) \) \(|h|\) units right if \( h > 0 \) and left if \( h < 0 \).

Second, graph \( g(x) = |x| + 2 \) and \( h(x) = |x| - 3 \).
You should have discovered that the graph of \( g(x) \) translated (shifted) vertically up 2 units the library absolute value function graph (\( f(x) = |x| \)) and that of \( h(x) \) translated \( f(x) \) down 3 units. \((1,1) \in f \Rightarrow (1,-1) \in g \) and \((1,4) \in h)\]

**In general** \( f(x) + k \) translates the graph of \( f(x) \) \(|k|\) units up if \( k > 0 \) and down if \( k < 0 \).

Third, graph \( g(x) = |2x| \) and \( h(x) = \frac{1}{3} |x| \).
You should have discovered that the graph of \( g(x) \) shrunk (compressed) horizontally by a factor of 2 the library absolute value function graph (\( f(x) = |x| \)) and that of \( h(x) \) stretched (expanded) \( f(x) \) horizontally by a factor of 3. \((1,1) \in f \Rightarrow (1/2,1) \in g \) and \((3,1) \in h)\]

**In general** \( f(ax) \) horizontally shrinks the graph of \( f(x) \) by a factor of \( a \) if \( a > 1 \) and stretches the graph of \( f \) by a factor of \( 1/a \) if \( 0 < a < 1 \).

Fourth, graph \( g(x) = 2|x| \) and \( h(x) = \frac{1}{3} |x| \).
You should have discovered that the graph of \( g(x) \) stretched (expanded) vertically by a factor of 2 the library absolute value function graph (\( f(x) = |x| \)) and that of \( h(x) \) shrunk (compressed) \( f(x) \) vertically by a factor of 3. \((1,1) \in f \Rightarrow (1,2) \in g \) and \((1,1/3) \in h)\]

**In general** \( bf(x) \) vertically stretches the graph of \( f(x) \) by a factor of \( b \) if \( b > 1 \) and shrinks the graph of \( f \) by a factor of \( 1/b \) if \( 0 < b < 1 \).
Finally, graph \( g(x) = \sqrt{-x} \) and \( h(x) = -\sqrt{x} \). You should have discovered that the graph of \( g(x) \) reflected (flipped) over the y-axis the library square root function graph (\( f(x) = \sqrt{x} \)), and that \( h(x) \) reflected \( f(x) \) over the x-axis. \((1,1) \in f \Rightarrow (-1,1) \in g \) and \((1,-1) \in h\]

In general \( f(-x) \) reflects the graph of \( f(x) \) over the y-axis and -\( f(x) \) reflects the graph of \( f \) over the x-axis

**Question:** Why do you think a different library function was used to develop this last transformation rule?

You have been examining transformations as operations that can be performed on a graph to make a new graph. As you might expect, many graphs are produced by doing multiple transformations. A natural question is, ”Does order matter?” The answer is yes. Just as there is an order of operations in performing computation; i.e. remove parentheses, apply exponents, do multiplication and division, and finally do addition and subtraction; there is an order in doing transformations. Simply put, the agreement is to do reflections and dilations (stretches and shrinks) first and then do translations.

Consider the function \( g(x) = 1 + 3\sqrt{-2x + 8} \). First rewrite the function as \( g(x) = 1 + 3\sqrt{-2(x - 4)} \).

Then take the square root graph \( f(x) = \sqrt{x} \) and reflect it over the y-axis, horizontally shrink it by 2, vertically stretch it by 3, shift it right 4 and up 1.

\[(1,1) \in f \Rightarrow (-1,1) \Rightarrow (-1/2,1) \Rightarrow (-1/2,3) \Rightarrow (7/2,3) \Rightarrow (7/2,4) \in g\]

**Exercise:** Using \( f \) and \( g \) above, transform the point \((-2,4) \in f \) to the corresponding point in \( g \) and then graph \( g(x) \).
LINE SYMMETRY: Two points are defined to be symmetric about a line if the
line is the perpendicular bisector of the line segment joining the points.

POINT SYMMETRY: Two points are defined to be symmetric about a point if the
point is the midpoint of the line segment joining the points.

Your response to the question, "Why do you think a different library function was
used to develop this last transformation rule?" seen earlier should have involved stating
that the graph of $f(x) = |x|$ when reflected over the y-axis would look the same. Graphs
that have this feature are said to have **Y-Axis Symmetry**. Functions or relations that
generate such graphs are said to be **even**. Algebraically $f(-x) = f(x)$, which means that if a
point $(a,b)$ belongs to this type of graph, then the point $(-a,b)$ does also.

In a similar manner, if $y = -f(x)$ produces the same graph as does $y = f(x)$, then the
graph has **X-Axis Symmetry**. Of course such a graph would fail the vertical line test and
thus would not be that of a function, but only a relation. If a point $(a,b)$ belongs to this
type of graph, then the point $(a,-b)$ does also.

Another type of symmetry, **Origin Symmetry**, occurs when $f(-x) = -f(x)$, which
means that if a point $(a,b)$ belongs to this type of graph then the point $(-a,-b)$ does also.
Functions or relations that generate a graph with this feature are called **odd**.

With regard to the labels of even and odd, these names come from **Polynomial
Functions** which will be studied in the next chapter. Polynomial Functions in one
variable take the form:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

where all the exponents are whole numbers. Now if all the exponents are even numbers,
the graph will have y-axis symmetry and if all the exponents are odd numbers, the graph
will have origin symmetry.

**Exercise:** Create some even and odd polynomial functions and graph
them using a grapher to convince yourself of the truth of the preceeding
statement.

**INVERSE RELATIONS**

Another type of symmetry important in mathematics is seen between **inverse
relations**. Two relations $f(x)$ and $g(x)$ are said to be inverses when a point $(a,b) \in f$ and
the corresponding point $(b,a) \in g$. The **graphs of inverse relations are symmetric to
each other about the line $y = x$**. This notion is best understood by examining the graphs
of a pair of inverse relations, $y = x^2$ and $x = y^2$. 


First observe that the line segment joining the pair of corresponding points (-1,1) and (1,-1) would have the line $y = x$ as a perpendicular bisector. This would be true for all pairs of corresponding points on the two graphs. Thus the graphs are symmetric about the graph of $y = x$. Second, observe that the points of intersection of the two inverse graphs [(0,0) and (1,1)] both lie on the graph of $y = x$. With some thought this second observation becomes obvious. The only way that $(a,b)$ and $(b,a)$ are the same point is for $a = b$, which is a point on the graph of $y = x$. Thus: A **second feature of inverse graphs is that if they intersect each other, they do so on the line $y = x$.**

In a later chapter you will study log and exponential functions as inverse functions. The inverse trigonometric functions, which have significant usage in calculus, are examined later (in the trig chapters) in this book. It will be useful to be able to determine if the inverse of a function is also a function and also how to find the inverse. First recall that a relation (defined as a set of ordered pairs $(x,y)$) is a function provided that for each $x$ there is one and only one $y$ value. Suppose that $a \neq c$ and two points $(a,b)$ and $(c,b)$ each belong to some function $f$. Then the points $(b,a)$ and $(b,c)$ would belong to the inverse of $f$, which would then mean the inverse of $f$ is not a function. Consequently, only functions for which there is a one-to-one matching between domain and range values (which are called **1 - 1 functions**) have inverses that are functions.

Now, we can begin the process of finding the inverse by realizing that the set of points that constitute the inverse are just those of the original function with the coordinates interchanged. Interchanging the coordinates can be accomplished by the simple act of interchanging the variables in the equation which generates the set of ordered pairs. For example, consider $y = f(x) = 2x - 3$. Interchanging $x$ and $y$ produces $x = 2y - 3$ which generates the points of the inverse. Now the easiest way to graph this new set of points is to solve this inverse generator equation for $y$, obtaining $y = \frac{1}{2} x + \frac{3}{2}$.

Borrowing the notation for the multiplicative inverse of a number (e.g. 2 and $2^{-1}$ are inverses), the inverse generator is denoted by $f^{-1}(x) = \frac{1}{2} x + \frac{3}{2}$. Although every function has an inverse (which may or may not be a function), it is often difficult or impossible to find. However, the symmetric relationship between a function and its inverse is important and useful in any case.

Previously you saw that, when working with composite functions, in general $f[g(x)]$ and $g[f(x)]$ are not the same function. However when using the functions from the preceding paragraph, we find that:

$$f[f^{-1}(x)] = 2\left[ \frac{1}{2} x + \frac{3}{2} \right] - 3 = x \quad \text{and} \quad f^{-1}[f(x)] = \frac{1}{2} \left[ 2x - 3 \right] + \frac{3}{2} = x$$
In general the composition of inverse functions always produces the identity function \( x \). This useful feature of inverse functions allows us a simple means of checking to make sure we found the inverse correctly.

**Problem Set 6.6**

For Problems 1 - 3, determine the domain of the function.

1. \( f(x) = 3x^2 - 2x + 4 \)
2. \( g(x) = 3 + \sqrt{x - 2} \)
3. \( h(x) = \frac{2}{x^2 - 3x - 10} \)

For Problems 4 - 9, let \( f(x) = 2x + 3; g(x) = \sqrt{x} \); and \( h(x) = \frac{1}{x^2} \).

4. Find: (a) \( f + g \)(x)   (b) \( g - h \)(x)  (c) \( fg \)(x)   (d) \( g \div h \)(x)
5. State the domains for the functions created in Problem 4.
6. Find: (a) \( f \circ g \)(x)   (b) \( g \circ f \)(x)  (c) \( h \circ g \)(x)   (d) \( g \circ g \)(x)
7. State the domains for the functions created in Problem 6.
8. Find (a) \( f^{-1} \)(x)   (b) \( g^{-1} \)(x)  (c) \( h^{-1} \)(x)
9. Which, if any, of the relations created in Problem 8 is **not** a function?

For Problems 10 - 12, with \( f(x) = x^3 \), identify the transformations on \( f \) and graph \( g(x) \).

10. \( G(x) = 2 - f(x + 3) \)
11. \( G(x) = 3 f \left( \frac{1}{2} x \right) \)
12. \( G(x) = \frac{1}{2} f(3x) \)

For Problems 13 - 16, identify any symmetry; identify as even, odd, or neither; and graph the relation.

13. \( y = x^4 \)
14. \( y = \pm \sqrt{x} \)
15. \( y = - x^3 \)
16. \( x^2 + y^2 = 4 \)
LAB 6.1 BATTLESHIP GAME ACTIVITY

DATE: ________________________________

INSTRUCTOR: ____________________________

MATERIALS NEEDED
Geoboards, Unifix Cubes

LAB GROUP MEMBERS:

Many of you may remember playing BATTLESHIP as a youth. This is a two person game in which players secretly place some number of ships on an "Ocean Grid" and then take turns "firing" at each other's ships until one player "sinks" all of the ships of the opponent. There are several versions of this game. The procedures and rules for the versions you will play are as follows:

GAME 1

Observe that the rows and columns on the geoboard are numbered 0 to 10. As an initial activity each of you place one "ship" (represented by 3 Unifix cubes placed on the pegs with the large opening up) either horizontally or vertically on your "Ocean Grid" which will be restricted to the lower left corner of the geoboard [point coordinates (0,0) to (5,5)].

Now take turns "firing" at your opponent's "ship" by calling out coordinates of a point on the grid. If the point named is covered by a cube representing part of your opponent's ship then a "hit" is recorded and you may "fire" again. If the "shot" is a miss then it becomes your opponent's turn to "fire." If your shot completes hits on all portions of your opponent's ship, then you win.

GAMES 2, 3, etc.

On your geoboard, place rubber bands representing horizontal and vertical axes intersecting near the center of the geoboard. Label (in your mind) the intersection point (0,0). The coordinates of any point on the grid will be (a,b) where a represents the horizontal movement (+ right, - left) and b the vertical movement (+ up, - down) from (0,0). For example, (-2,3) is located left 2 and up 3 from (0,0).

Note: You will not be using the numbers marked on the geoboard for these games.

Without your opponent seeing, place your ships on your geoboard by covering adjacent pegs (horizontally, vertically, or diagonally as you agree with your opponent). A "Carrier" will be represented by 5 cubes, a "Battleship" by 4 cubes, a "Cruiser" by 3 cubes, and "Subs" or "Destroyers" by 2 cubes. Players are to agree on the size of their "Fleet" and "Ocean", but obviously for a fair "WAR" opponents should have the same total number of cubes on the same size grid. When all the cubes representing one of your ships are hit you are to say "You have sunk my-- (name of ship)." The game continues until the "WINNER" sinks all the opponent's ships.
After playing BATTLESHIP a few times, write a reaction to this game and indicate some useful strategies you discovered.
LAB 6.2  COORDINATE GEOMETRY ACTIVITY

DATE: ________________________________

INSTRUCTOR: __________________________

LAB GROUP MEMBERS:

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__________________________

MATERIALS NEEDED
None

1. Point M is 3 units left and 5 units up from the origin and thus has coordinates (-3,5). On the grid above locate and label the following points: A(2,5), T(4,-3), H(-1,-3).

2. Using the points plotted above, draw polygon MATH.

3. What type of polygon is MATH? Explain.

4. What is the perimeter of MATH?

5. What is the area enclosed by polygon MATH?

6. Suppose quadrilateral JOYS is formed by joining in order the midpoints of the sides of MATH. What type of quadrilateral is JOYS? Explain.
II. On the grid below, plot points A (-6, -4), B (6, 1), C (3, 5), and D (-1, 8). Find the perimeter and area of quadrilateral ABCD. [Hint: Box in the quadrilateral with a rectangle having horizontal and vertical sides.]
LAB 6.3  PARALLEL AND PERPENDICULAR LINES

DATE: ___________________________  LAB GROUP MEMBERS: ___________________________

INSTRUCTOR: ______________________

MATERIALS NEEDED
Protractor

In this investigation, you will discover the interesting relationship that exists between the slopes of (I) parallel lines and (II) perpendicular lines.

I. PARALLEL LINES

Two coplanar lines which do not intersect are called parallel. Complete steps 1-6 to discover the relationship between the slopes of parallel lines.

1. Attempt to solve each of the following systems of linear equations.
   
   (a) \(2x + 3y = 5\) \(\quad\) (b) \(2x - 3y = 12\) \(\quad\) (c) \(3x - 9y = 8\)
   
   \(2x + 3y = 7\) \(\quad\) \(-4x + 6y = 20\) \(\quad\) \(-2x + 6y = 13\)

2. What did you discover in your solution attempts?

3. What can be said about the solution of each of the pairs of lines?

4. Write each of the pairs of lines in Problem 1 in slope-intercept form.

5. What do you observe concerning the slopes of each pair of lines?

6. Write a conjecture concerning the slopes of parallel lines. To prove your conjecture, consider the figure below and complete the following steps.
7. Why does \( m_1 = \frac{b_1}{a_1} \) and \( m_2 = \frac{b_2}{a_2} \) ?

8. Why is \( \angle B = \angle E \) ?

9. Consequently, why is \( \triangle ABC \sim \triangle DEF \)? [Hint: recall the 3 methods of showing triangles similar seen in Chapter 8.]

10. Therefore, how are \( \angle 1 \) and \( \angle 2 \) related?

11. Thus, how are \( \angle 3 \) and \( \angle 4 \) related?

12. Thus, how are \( L_1 \) and \( L_2 \) related?

Conversely, it is also easy to prove that if \( L_1 \parallel L_2 \), then \( m_1 = m_2 \)
II. PERPENDICULAR LINES

Perpendicular lines intersect at right angles.

1. Consider the family of lines shown below. Using your protractor, determine which of the pairs of lines are perpendicular and write as number pairs the slopes of the perpendicular lines.

2. Suppose a line had equation \( y = 5x + 1 \). What would be the equation of the line perpendicular to this line and having the same y intercept?

3. Recall that for a non-zero number \( x \), the number \( \frac{1}{x} \) is called the reciprocal of \( x \) and the number, \( -x \), is called the negative of \( x \). The number \( -\frac{1}{x} \) is called the negative reciprocal of \( x \). Using this terminology, state in words a conjecture for the relationship between the slopes of non-horizontal or non-vertical perpendicular lines.

To prove your conjecture, consider the figure on the next page and complete the following steps.
Given: $L_1 \perp L_2$, $m_1 = \frac{DE}{CD}$, $m_2 = \frac{-AB}{BC}$  
Prove: $m_1 = -\frac{1}{m_2}$

4. Why is the task of proving $m_1 = -\frac{1}{m_2}$ equivalent to that of proving $\frac{DE}{CD} = \frac{BC}{AB}$?

5. Since $L_1 \perp L_2$, what is $m \angle ACE$?

6. Consequently, how are $\angle 1$ and $\angle 2$ related?

7. Now since $m \angle ABC = 90^\circ$, how are $\angle 1$ and $\angle 3$ related?

8. Thus, how are $\angle 3$ and $\angle 2$ related?

9. Why then is $\triangle ABC \sim \triangle CDE$?

10. Complete the proof to show $m_1 = -\frac{1}{m_2}$.

Conversely, it is also easy to prove that if $m_1 = -\frac{1}{m_2}$, then $L_1 \perp L_2$. 

**Lab 6.4 Introduction to Matrices**
SOLVING LINEAR SYSTEMS WITH MATRICES

Consider the following system along with its solution by elimination and substitution.

\[
\begin{align*}
\begin{cases}
x - y + z &= 6 \\
3x + 2y - z &= -4 \\
-2x + y + 2z &= 2
\end{cases}
\Rightarrow
\begin{cases}
x - y + z &= 6 \\
-3E_1 + E_2 &= 5y - 4z = -22 \\
2E_1 + E_3 &= -y + 4z = 14
\end{cases}
\Rightarrow
\begin{cases}
x - y + z &= 6 \\
y + 12z = 34 \\
- y + 4z = 14
\end{cases}
\Rightarrow
\begin{cases}
x - y + z &= 6 \\
y + 12z = 34 \\
16z = 48
\end{cases}
\end{align*}
\]

Now using back substitution, \(z = 3\); \(y = -2\) and \(x = 1\).

Observe that the variables did not really aid the solution; they just occupied space. A shorter solution technique eliminating writing the variables and using parentheses is shown below.

\[
\begin{align*}
\begin{pmatrix}
1 & -1 & 1 & 6 \\
3 & 2 & -1 & -4 \\
-2 & 1 & 2 & 2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & -1 & 1 & 6 \\
5 & -4 & -22 \\
-1 & 4 & 14
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 12 & 34 \\
1 & 12 & 34
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 12 & 34 \\
16 & 48
\end{pmatrix}
\end{align*}
\]

As before, using back substitution produces \(z = 3\); \(y = -2\) and \(x = 1\).

Rectangular arrays of numbers such as seen above are called Matrices (singular – Matrix). When solving systems using matrices, you simply add multiples of rows to each other until you produce a form of a matrix which allows you to solve for the variables by back substitution. Observe that the initial matrix consists of a square matrix of the coefficients augmented by the column matrix of the constants. Thus the technique used to solve the system in the manner seen above is often called the Augmented Matrix method.

1. Solve the following system using the augmented matrix method.

\[W + X + Y + Z = 2\]
THE ALGEBRA OF MATRICES

Matrices have significant utility beyond notational convenience. Much of this utility depends upon agreement on the algebra for matrices.

**Dimension** - Matrices are distinguished by their dimensions. An \( m \times n \) matrix has \( m \) rows and \( n \) columns.

**Equality** - Matrices are considered to be equal if they are identical. This means that they have the same dimensions and the same elements in the same positions.

**Addition** – Matrices are added by adding together numbers in the same position. This procedure necessitates that two matrices are compatible for addition if and only if they have the same dimension.

Example:

\[
\begin{pmatrix}
1 & -2 & 3 \\
-4 & 5 & -6
\end{pmatrix}
+ 
\begin{pmatrix}
2 & 2 & -4 \\
3 & -5 & 1
\end{pmatrix}
= 
\begin{pmatrix}
3 & 0 & -1 \\
-1 & 0 & -5
\end{pmatrix}
\]

Subtraction – Subtraction is performed by changing the signs of the numbers in the matrix to be subtracted and then adding.

**Multiplication** – A product \( AB \), is obtained by multiplying the numbers in each row of \( A \) times the corresponding numbers in each column of \( B \) and adding up these products to produce an entry in the product matrix.

Example:

\[
\begin{pmatrix}
a & b & c \\
d & e & f
\end{pmatrix}
\begin{pmatrix}
g & h \\
i & j \\
k & l
\end{pmatrix}
= 
\begin{pmatrix}
ag+bi+ck & ah+bj+cl \\
dg+ei+fk & dh+ej+fl
\end{pmatrix}
\]

Now you should realize that this procedure necessitates that two matrices are compatible for multiplication if and only if the second dimension (# columns) of matrix A equals the first dimension (#rows) of matrix B. For the example a 2X3 matrix times a 3X2 matrix produced a 2X2 matrix. In general a \( m \times n \) matrix times
a $n \times k$ matrix produces a $m \times k$ matrix. You should also realize that this definition means that even if both products $AB$ and $BA$ exist for two matrices, $A$ and $B$, these products need not be the same. (The commutative law for multiplication does not hold for matrices.) Later in this lab you will see why this particular method of multiplying matrices is utilized.

**Division** – Division of some matrices is accomplished multiplying by the inverse, as we will investigate later in this lab.

**Powers** – As with simple numbers, the $n^{th}$ power of a matrix means multiply it times itself $n$ times. You should realize that this means you can only raise square matrices (the dimensions are the same) to a power.

2. Let $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & 0 \\ 4 & -3 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -3 \\ 1 & 5 \\ -4 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$.

Find or state why it does not exist: (a) $A + B$ (b) $AB$ (c) $BC$ (d) $D^3$ (e) $B^2$

3. The two main flavors of Clemson ice cream are chocolate and vanilla. The following table shows the number of gallons of each flavor sold per week at three different locations. Profit per gallon on Vanilla is $4 and on chocolate is $3. Use matrices to find the total profit on each flavor at each location.

<table>
<thead>
<tr>
<th></th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chocolate</td>
<td>100</td>
<td>80</td>
<td>120</td>
</tr>
<tr>
<td>Vanilla</td>
<td>160</td>
<td>120</td>
<td>100</td>
</tr>
</tbody>
</table>

**MATRIX EQUATIONS**

The preceding computational definitions allow us to solve systems of equations in yet another manner. Consider the system seen earlier.
This system can be expressed as a matrix equation as follows.

\[
\begin{pmatrix}
1 & -1 & 1 \\
3 & 2 & -1 \\
-2 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
6 \\
-4 \\
2
\end{pmatrix}
\]

If we denote the coefficient matrix as \(A\), the variable matrix as \(X\) and the constant matrix as \(K\), we have the equation \(AX = K\). Multiplying through by the inverse of \(A\), indicated by \(A^{-1}\), produces \(A^{-1}AX = A^{-1}K\). Now the inverse of a matrix such as \(A\) is defined to be such that \(A^{-1}A = AA^{-1} = I\) where \(I\) (which is called an identity matrix— you will see why shortly) is a matrix having 1’s in the main diagonal (upper left to lower right) and 0’s everywhere else. Thus we have the equation \(IX = A^{-1}K\). Observe that

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
.  \{You see why I is called the identity matrix.\]

Hence \(X = A^{-1}K\)

Now it is tedious to calculate the inverse of a matrix by hand, but a calculator or computer can do this easily.

\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 1 \\
3 & 2 & -1 \\
-2 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

by creating and solving systems of equations. Then find \(A^{-1}K\) to solve for \(X\).

**MARKOV CHAINS**

Consider the following activity: Three kids are observed throwing a ball. Anna always throws the ball to Ben and Ben always throws the ball to Carol; but Carol is just as likely to throw the ball to Anna as to Ben. This information is tabulated below.
As a matrix, we have
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\]

As a matrix, we have
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

Note: The rows of this matrix can be thought of as probability vectors (all non-negative entries and having a sum of one), so you see a relationship between vectors and matrices. A matrix of this type is called a stochastic matrix.

Now suppose Anna has the ball. The initial state for this activity can be represented by the probability vector \( (1 \ 0 \ 0) \). The matrix seen above is called the transition matrix for this activity
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\]

After one toss we have \( (1 \ 0 \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (0 \ 1 \ 0) \) [Ben has the ball.]

State two is obtained by multiplying State one times the transition matrix T. Symbolically we write \( S_2 = S_1 \times T \)
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

After two tosses we have \( (0 \ 1 \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (0 \ 0 \ 1) \) [Carol has the ball.]

\( S_3 = S_2 \times T \)

This process, which is called a Markov Chain can be continued and recursively we have \( S_{k+1} = S_k \times T \).

Note: It can be shown that all subsequent states are also probability vectors.

5. Using the activity described above, show that subsequent states could be alternatively found by multiplying the initial state times powers of the transition matrix i.e. \( S_3 = S_1 \times T^2 \), \( S_4 = S_1 \times T^3 \), etc. Then state a conjecture for a formula for the \( n^{th} \) state in the space below.

A natural question which occurs at this time is, “If this activity continues, would the probabilities concerning who has the ball approach fixed values?” Another way to ask this question is, “Is there a steady state?”

Let the probability vector \( (x \ y \ z) \) represent the alleged steady state. Then
(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z) \text{ or } (\frac{1}{2}z, x + \frac{1}{2}z, y) = (x, y, z)

Writing equations, we have $\frac{1}{2}z = x, x + \frac{1}{2}z = y, and y = z$. Since $(x, y, z)$ is a probability vector, we also have that $x + y + z = 1$. As a simplified system we have

\begin{align*}
2x - z &= 0 \\
2x - 2y + z &= 0 \\
y - z &= 0 \\
x + y + z &= 1
\end{align*}

6. Solve the system above to verify that the steady state for the ball tossing activity is $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ and interpret what this means.

Use Markov Chains to solve the two problems on the next page.

7. **Study Habits:** If a student studies one night then he or she is 70% sure not to study the next night. Additionally, the probability of not studying two successive nights is 60%. In the long run, how often does this student study?
8. Many, many years ago three companies, Alpha, Beta and Gamma simultaneously introduced the same product. Initially Alpha had 40% of the market; Beta 20% and Gamma 40%. After one year, Alpha retained 85% of its customers, took 15% of Beta’s customers and took 5% of Gamma’s customers. Beta retained 75% of its customers, took 5% of Alpha’s customers and took 5% of Gamma’s customers. Gamma retained 90% of its customers, took 10% of Alpha’s customers and 10% of Beta’s customers. If this pattern of customer retainment/recruitment has continued throughout time, what percentages of the market does each company have now?
Businesses are interested both in maximizing profit and in minimizing costs. Transportation companies are interested in having the most efficient delivery route. During World War II a process known as linear programming was developed to facilitate the efficient transportation of troops and supplies to and from the various operation theaters. The purpose of this relatively new application of mathematics was to maximize or minimize a function of several variables subject to a set of constraints which take the form of linear inequalities. Although most linear programming problems are very complex and necessitate utilization of a computer to effect a solution, some problems can be solved via a graphical approach. You will be solving such a problem in this lab activity.

The Nrek School District plans to have the instructional staff of Music City West Elementary School composed of credentialed teachers and non-credentialed teacher aides. Teachers earn an average of $25000 per year and the aids earn $10000 per year. The district wants to minimize the salary costs involved but must adhere to the guidelines stated below:

- The instructional staff must number at least 20, but no more than 35.
- The teachers must number at least 15.
- The number of aids can not exceed twice the number of teachers.

To solve this linear programming problem, complete the steps below.

1. Begin by assigning the variables.
   - Let \( x = \) number of teachers \( y = \) number of teacher aides.
   - (a) Now translate, "The instructional staff must number at least 20."
     Your translation should be an inequality in \( x \) and \( y \).
   - (b) Translate, "The instructional staff must number no more than 35."
     Your translation again should be an inequality in \( x \) and \( y \).
   - (c) Translate, "The teachers must number at least 15."
     Your translation should be an inequality just in \( x \).
   - (d) Translate, "The number of aids can not exceed twice the number of teachers."
     Your translation should again be an inequality in \( x \) and \( y \).
2. Write two inequalities which indicate that it is not possible to have a negative number of teachers or aids.

3. These six inequalities establish the system of inequalities for this problem. List the six inequalities here:

This system of inequalities represents the constraints of the problem. You are interested in finding the number of teachers and aids which will satisfy all six of these constraints. Locating possible solutions involves graphing all six of these inequalities in the same coordinate plane.

4. Use the grid pictured below to graph all six inequalities. Graphing a linear inequality requires these steps:
   i. Graph the line as if the inequality symbol were an equal symbol. Draw the line solid if the inequality is \( \leq \) or \( \geq \), and dashed if it is \( < \) or \( > \).

   ii. Pick a test-point in the coordinate plane that does not lie on the line. Substitute the coordinates of the test point into the inequality. If the result is true, shade the half-plane which contains the test-point. If the result is false, shade the opposite half-plane.
Chapter 6 Lab Activities

The region of the graph which is shaded for all six inequalities is called the **feasible region**, or the **region of feasible solutions**. Any point in the feasible region is a solution to the system. You can verify this by choosing points in the feasible region and testing each point in all of the inequalities of the system. Actually, there are an infinite number of feasible solutions. However, you are interested in the best, or optimum, solution to the problem.

5. In the original problem, what do you wish to minimize?

6. Write an algebraic expression which represents the quantity in question 5.

**Theorem:** If an optimum value (either a maximum or a minimum) to a linear programming problem exists, it will occur at one or more of the **vertices** of the feasible region.

According to the theorem, the best solution to this problem is located at one or more of the corners of the feasible region.

7. List the coordinates (both on your graph and in the space below) of the vertices of the feasible region. You may be able to read the coordinates from your graph. If not, you must find the coordinates algebraically.

8. Which coordinate pair is the best solution? Why?

This completes the process of solving a linear programming problem by graphing inequalities.
AN APPLICATION OF PARABOLAS, CIRCLES, AND ELLIPSES

1. **The Parabola** Recall that a quadratic equation, \( y = ax^2 + bx + c \) can be written in the form \( y = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \) by completing the square. The graph is a parabola which opens up if \( a > 0 \) and down if \( a < 0 \). The vertex is at \( \left( -\frac{b}{2a}, \frac{-b^2 + 4ac}{4a} \right) \). [You should verify this if you have forgotten it!]

(a) In general, what must be true concerning \( a, b, \) and \( c \) if a certain parabola looks as follows? [Hint: The vertex is the y-intercept.]

![Parabola Graph](image)

(b) If the x intercepts of the parabola are 25 and -25, and the y intercept (the vertex) is 20, write the equation and sketch the graph.
2. **The Circle** Recall that a circle is a set of points in a plane equidistant from a given fixed point called the center.
   
   (a) Label the intercepts for the circle with center at the origin sketched below.

   ![Circle Diagram]

   (b) Use the distance formula and the fact that $OP = r$ to show the equation of the circle is $x^2 + y^2 = r^2$.

   (c) If $r = 25$, write the equation of the circle and sketch the graph.

3. **The Ellipse** Recall that an ellipse can be thought of as a flattened circle. A standard ellipse with center at the origin has an equation of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and looks as follows:

   ![Ellipse Diagram]

   (a) Using $a$ and $b$ place the coordinates of the intercepts on the graph above.

   (b) The major (longer) axis (segment containing the center which joins a pair of intercepts) appears to be horizontal. What is its length (in terms of $a$ or $b$).

   (c) What is the length of the minor (shorter) axis?
(d) What is the equation of the ellipse shown below?

4. **The Tunnel** A tunnel is to be drilled through a mountain to build a road 50 feet wide.

(a) If the tunnel is in the shape of a semicircle, how high will it be in the center?

(b) If the tunnel needs only to accommodate vehicles 15 feet high, why would a parabolic arch or a semi-ellipse be a better choice than the circle for the shape of the tunnel? Draw pictures and explain.

(c) Suppose the height of the tunnel is 20 feet. Determine how far from the center line could a 15 foot high vehicle travel through the tunnel without hitting the ceiling if the tunnel is:
   (i) a parabolic arch and
   (ii) a semi elliptical arch.

Refer to problems 1b and 3d, draw pictures and show your calculations.

(d) Which of the three shapes is best for building a tunnel? Why?
6.1 GSP: SLOPES OF SPECIAL LINES

A. In the GRAPH menu, choose SHOW GRID. Construct a line and measure its slope. Drag the line and observe changes to the slope. Construct the intersection of the line and the y-axis. Measure its coordinates. Measure the line’s equation and determine the relationship between this point and the equation and between the slope and the equation. What do you note about the slopes of vertical and horizontal lines? State your observation in the form of a conjecture.

B. Construct parallel lines and investigate the relationship between their slopes. State your observations in the form of a conjecture.

C. Investigate the relationship between the slopes of two perpendicular lines. Since nothing may be readily apparent from the measures of the slopes, use MEASURE, CALCULATE to investigate their product. State your observations in the form of a conjecture.
6.2 GSP: DISTANCE TO THE LIGHTHOUSE

DATE: ___________________________  LAB GROUP MEMBERS: ___________________________

INSTRUCTOR: _____________________  ___________________________

MATERIALS NEEDED  
Geometer’s Sketchpad

You are on a ship traveling at 12 miles per hour. At 2100 hours the lookout on the mast spots the signal of a lighthouse at $15^\circ$ to port (left). Six hours later the signal of the lighthouse has moved to $145^\circ$ to port. In the sketch below, the ship is moving up. Your challenge is to set up a scale drawing on Sketchpad and determine how close the ship came to the lighthouse.
6.3 GSP: MIDPOINTS AND MIDSEGMENTS

DATE: ________________________________

LAB GROUP MEMBERS:

INSTRUCTOR: _________________________

MATERIALS NEEDED
Geometer’s Sketchpad

A. Construct a segment. Construct its midpoint. Select the endpoints of the segment and the midpoint and measure their abscissas (their x-values) and their ordinates (their y-values). Make a conjecture about the relationship between the abscissas of the endpoints and the abscissa of the midpoint and between the ordinates of the endpoints and the ordinate of the midpoint. As always, drag your sketch to see if your conjecture holds.

B. Measure the A midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. What is the relationship between the length of the midsegment and the base of the triangle? After you determine this relationship, explain why you think this relationship exists.

C. Determine the relationship between the slope of the midsegment and the slope of the base.

D. What is the relationship between the perimeter of the triangle formed by a midsegment and the perimeter of the larger triangle? Why do you think this relationship exists?
E. What is the relationship between the area of the triangle formed by a midsegment and the area of the larger triangle? Why do you think this relationship exists?